5. HARMONIC ANALYSIS ON X AND INVERSION OF R

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5. Harmonic analysis on X and inversion of R

As noted in Section 3.2, an inversion formula for the Radon transform on $X = G/K$ follows from a convolution inverse T of the distribution S in Propositions 3 or 4: if $S*T = \delta$, then $u = (R^*Ru)*T$. Since S is K-invariant it is natural to search for a K-invariant T by means of harmonic analysis of radial functions on X, i.e. solving the equation

$$\widetilde{S}(\lambda)\,\widetilde{T}(\lambda) = 1$$

where $\widetilde{\ }$ denotes the spherical transform. We keep to the notations of Section 4.1 **b**, dealing with the *d-geodesic transform on a n-dimensional hyperbolic space X.*

The spherical function φ_λ on X is the radial eigenfunction of the Laplacian L defined by

$$L\varphi_\lambda = -\left(\lambda^2 + \rho^2\right)\varphi_\lambda\,, \quad \varphi_\lambda(x_o) = 1\,,$$

where $\rho = (p/2) + q$ and λ is a real parameter. Writing down the radial part of L it follows that ([9], p. 484)

$$\varphi_\lambda(r) = {}_2F_1\left(\frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}; \frac{n}{2}; -\sinh^2 r\right), \tag{8}$$

where ${}_2F_1$ is the classical hypergeometric function. The spherical transform of a radial function S is then

$$\widetilde{S}(\lambda) = \int_X \varphi_\lambda(x) S(x)\,dx = \int_0^\infty \varphi_\lambda(r) S(r) A(r)\,dr \tag{9}$$

and, in view of the relevant expressions (4) and (5) of A and S (Section 4.1 with $\varepsilon = 1$), we shall need the following lemma.

LEMMA 10. *Let $a, b, \alpha, \beta, \gamma$ be complex numbers, with $0 < \operatorname{Re} a < \operatorname{Re}\gamma$, $\operatorname{Re}(a+b) < \operatorname{Re}\alpha$ and $\operatorname{Re}(a+b) < \operatorname{Re}\beta$. Then*

$$\int_0^\infty {}_2F_1(\alpha, \beta; \gamma; -s)\, s^{a-1}(1+s)^b ds$$
$$= \frac{\Gamma(\gamma)\Gamma(a)}{\Gamma(\gamma - a)} \frac{\Gamma(\alpha - a - b)\Gamma(\beta - a - b)}{\Gamma(\alpha - b)\Gamma(\beta - b)}\, {}_3F_2(a, -b, \alpha + \beta - \gamma - b; \alpha - b, \beta - b; 1)\,.$$

Here ${}_3F_2$ denotes the generalized hypergeometric series

$${}_3F_2(a, b, c; d, e; z) = \sum_{n \geq 0} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{z^n}{n!}\,,$$

with $(a)_o = 1$, $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$. The lemma follows from the change $s = t/(1-t)$, some classical identities for ${}_2F_1$ and term by term integration under $\int_0^1 (\ldots)\,dt$ of the power series expansion of ${}_2F_1$. We skip the details of the proof. Among various equivalent expressions which could be obtained similarly, the above one was chosen because of its obvious symmetry with respect to α and β.

Let $\mu = \left(1 - q' + \rho + i\lambda\right)/2$. Changing u into $-\sinh^2 r$ in Lemma 10 we obtain, in view of (4), (5), (8) and (9),

$$(10) \qquad \widetilde{S}(\lambda) = \frac{\pi^{d/2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-d}{2}\right)} \left|\frac{\Gamma\left(\mu - \frac{d}{2}\right)}{\Gamma(\mu)}\right|^2 \cdot {}_3F_2\left(\frac{d}{2}, \frac{1-q'}{2}, \frac{q-q'}{2}; \mu, \overline{\mu}; 1\right),$$

assuming $\lambda \in \mathbf{R}$, $0 < d < n$ and $d < 1 - q' + \rho$; recall that $q = \dim \mathfrak{p}_{2\alpha}$, $q' = \dim \mathfrak{s}_{2\alpha}$.

Finding T such that $\widetilde{S}(\lambda)\widetilde{T}(\lambda) = 1$ seems intractable however, unless the ${}_3F_2$ factor is trivial. We are thus led to assume from now on (as in Section 4.2)

$$(11) \qquad q' = q \quad \text{or} \quad q' = 1,$$

so that ${}_3F_2(\cdots) = 1$. The conditions on d then reduce to $0 < d < 1 - q' + \rho$, ensuring the convergence of the integral $\widetilde{S}(\lambda)$.

If $d = 2k$ is an *even* integer, then $\widetilde{S}(\lambda)$ is the reciprocal of the polynomial

$$\frac{1}{\widetilde{S}(\lambda)} = C \prod_{j=1}^{k} \left(-\lambda^2 - \rho^2 + (n - 2j - q' + q)(2j + q' - 1)\right),$$

where C is a constant factor, in agreement with Theorem 8 (with $-\lambda^2 - \rho^2$ corresponding to L under the spherical transform).

If $d = p' + q' + 1$ is *odd*, then $p' = \dim \mathfrak{s}_\alpha$ and $q' = \dim \mathfrak{s}_{2\alpha}$ must have the same parity which, according to the classification of rank one symmetric spaces, can only occur for $q' = 0$ and p' even. Condition (11) now requires $q = 0$, and X must be a real hyperbolic space $H^n(\mathbf{R})$. This is the case studied by Berenstein and Tarabusi [1]; see also [14] p. 101 for $X = H^2(\mathbf{R})$ and $d = 1$. To find the convolution inverse T the strategy is to consider

$$f_{a,b}(r) = (\sinh r)^a (\cosh r)^b,$$

where a, b are chosen so that $\widetilde{f_{a,b}}(\lambda)$ has (by Lemma 10 again) an expression similar to (10) with trivial hypergeometric factor, and so that cross simplifications occur between the $|\Gamma(\cdots)|^2$ factors in the product $\widetilde{S}(\lambda)\widetilde{f_{a,b}}(\lambda)$. This product is then the reciprocal of a polynomial in λ^2 (as in the case d even), and the corresponding inversion formula is

$$u = P(L)\left((R^*Ru) * f_{a,b}\right) ,$$

where P is a polynomial. We refer to [1] for details.

Unfortunately the method of spherical transforms sketched above seems to provide explicit inversion formulas for the d-geodesic Radon transform on X only when $q' = q$ or $q' = 1$ on the one hand (to get rid of ${}_3F_2$) *and* d even or $X = H^n(\mathbf{R})$ on the other hand. The only reachable results so far are thus the formulas already obtained in [1] for $H^n(\mathbf{R})$ and a new proof of the above Theorem 8. The method might however yield some new results in the wider class of Damek-Ricci spaces (or harmonic NA groups), where the dimension q can be an arbitrary integer.

6. Shifted Radon transforms, waves, and the amusing formula

On page 146 of [10], S. Helgason notes the "amusing formula"

$$LR^*Ru(x) = -\frac{\partial}{\partial\tau}R^*_{t(\tau)}Ru(x)\Big|_{\tau=1} \tag{12}$$

for the 2-geodesic Radon transform R on the hyperbolic space $X = H^3(\mathbf{R})$, where L is the Laplace-Beltrami operator of X and $x \in X$. In this formula, R^*_t is the generalized dual transform obtained by integrating over all 2-dimensional totally geodesic submanifolds *at distance* t from a point x, and $t = t(\tau)$ denotes the positive solution of the equation $\cosh t = 1/\tau$. In [10], or [11], p. 55, equation (12) is indirectly obtained by bringing together two different inversion formulas for R.

In this section we study general shifted transforms, a concept going back to Radon himself [16] for the line transform in $\mathbf{R}^2$, and we use them to derive inversion formulas. They also provide solutions of wave-type equations; formula (12) can actually be seen as a wave equation at time $t = 0$. We shall give a direct proof of some generalized "amusing formulas", thus solving wave equations (called multitemporal when the time variable is multidimensional), and we use them to relate two different types of Radon inversion formulas (with or without shifts).