

# 6.5 MULTITEMPORAL WAVES

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scalar product  $(T, V) = \operatorname{Re} \sum \bar{T}_i V_i$  on  $\mathfrak{p}$  we have  $(T, V\lambda) = (T\bar{\lambda}, V)$ ,  $\lambda \in \mathbf{F}$ , therefore  $\mathfrak{s}^\perp$  is a  $\mathbf{F}$ -subspace of  $\mathfrak{p}$ .

An element  $k$  of  $K \cap H$  is characterized by  $k \in K$  and  $k \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$ , i.e.  $k \cdot \mathfrak{s} = \mathfrak{s}$  (adjoint action). Let  $n'$ ,  $d'$  be the respective dimensions of  $\mathfrak{p}$  and  $\mathfrak{s}$  as  $\mathbf{F}$ -vector spaces. Taking a  $\mathbf{F}$ -basis of  $\mathfrak{p}$  according to the decomposition  $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ , it follows that

$$K = U(1; \mathbf{F}) \times U(n'; \mathbf{F}), \quad K \cap H = U(1; \mathbf{F}) \times U(d'; \mathbf{F}) \times U(n' - d'; \mathbf{F}).$$

But  $U(n' - d'; \mathbf{F})$  acts transitively on the unit sphere of  $\mathbf{F}^{n' - d'}$ , which implies our claim.

If  $T, T' \in \mathfrak{s}^\perp$  are two unit vectors, there exists  $k_o \in K \cap H$  such that  $k_o \cdot T = T'$ . Thus

$$\begin{aligned} R_{\exp tT'}^* v(gK) &= \int_K v(gkk_o \exp(tT)k_o^{-1}H) \, dk \\ &= \int_K v(gk \exp(tT)H) \, dk = R_{\exp tT}^* v(gK). \end{aligned}$$

In particular  $R_{\exp tT}^* v$  is an even function of  $t$ .

Going back to (23), we now take as  $(X_j)$  an orthonormal  $\mathbf{R}$ -basis of  $\mathfrak{p}$  according to the decomposition  $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ . The  $n - d$  basis vectors in  $\mathfrak{s}^\perp$  give the same contribution to the right hand side, whereas the  $d$  vectors in  $\mathfrak{s}$  generate one parameters subgroups of  $H$  and give no contribution; indeed  $\exp tV \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$  for  $V \in \mathfrak{s}$ , since  $\mathfrak{s}$  is a Lie triple system by Section 4.3 c. This completes the proof.  $\square$

## 6.5 MULTITEMPORAL WAVES

We shall now deal with general invariant differential operators. As before  $G$  is a Lie group,  $H$  a closed subgroup,  $K$  a compact subgroup, and  $X = G/K$ ,  $Y = G/H$ . Let  $\mathfrak{g}$ ,  $\mathfrak{h}$ ,  $\mathfrak{k}$  be the respective Lie algebras, and  $\mathfrak{t}$  a vector subspace of  $\mathfrak{g}$  such that

$$\mathfrak{g} = (\mathfrak{k} + \mathfrak{h}) \oplus \mathfrak{t}.$$

Let  $K_1, \dots, K_p$  be a basis of  $\mathfrak{k}$ , complemented by  $H_1, \dots, H_q \in \mathfrak{h}$  so that the  $K_i$ 's and  $H_j$ 's are a basis of  $\mathfrak{k} + \mathfrak{h}$ , and let  $T_1, \dots, T_r$  be a basis of  $\mathfrak{t}$ . We shall use the same notations for the corresponding left-invariant vector fields on  $G$ , e.g.

$$K_i f(g) = \partial_s f(g \exp sK_i)|_{s=0},$$

with  $f \in C^\infty(G)$ ,  $g \in G$ ,  $s \in \mathbf{R}$ . We denote by  $\mathbf{D}(G)$  the algebra of all left invariant differential operators on  $G$ , by  $\mathbf{D}(G)^K$  the subalgebra of right

$K$ -invariant operators and by  $\mathbf{D}(X)$  the algebra of  $G$ -invariant differential operators on  $X$ . For  $s = (s_1, \dots, s_r) \in \mathbf{R}^r$ , let

$$t(s) = \exp s_1 T_1 \cdots \exp s_r T_r.$$

We recall that, for  $g, t \in G$ ,

$$R_t^* v(gK) = \int_K v(gktH) dk.$$

**THEOREM 17.** *Let  $G$  be a Lie group,  $H, K$  Lie subgroups, with  $K$  compact and  $X = G/K$ ,  $Y = G/H$ .*

(i) *For any  $P \in \mathbf{D}(X)$  there exists  $Q(\partial)$ , a constant coefficients differential operator on  $\mathbf{R}^r$ , with  $\text{order}(Q) \leq \text{order}(P)$ , such that for any  $v \in C^\infty(Y)$ ,  $x \in X$ ,*

$$(25) \quad PR^* v(x) = Q(\partial_s) R_{t(s)}^* v(x) \Big|_{s=0}.$$

(ii) *Assume furthermore that  $\mathfrak{t}$  is a Lie subalgebra of  $\mathfrak{g}$  with  $[\mathfrak{t}, \mathfrak{h}] \subset \mathfrak{h}$ , and let  $T$  denote the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{t}$ . Then for any  $P \in \mathbf{D}(X)$  there exists a right-invariant differential operator  $Q$  on  $T$ , with  $\text{order}(Q) \leq \text{order}(P)$ , such that*

$$(26) \quad P_{(x)} R_t^* v(x) = Q_{(t)} R_t^* v(x)$$

for  $v \in C^\infty(Y)$ ; here  $P_{(x)}$  acts on the variable  $x \in X$  and  $Q_{(t)}$  acts on  $t \in T$ .

Thus  $R_t^* v(x)$ , as a function of  $(x, t) \in X \times T$ , solves the generalized “multitemporal” wave equation (26) with time variable in a multidimensional space. Similarly (25) can be viewed as a wave equation in the variables  $(x, s) \in X \times \mathbf{R}^r$  at the time  $s = 0$ .

*Proof.* In order to work on  $G$  rather than on its homogeneous spaces, we define  $w(g) = v(gH)$  and, for  $g, t \in G$ ,

$$(27) \quad F(g, t) = (R_t^* v)(gK) = \int_K w(gkt) dk,$$

so that  $F(gk, k'th) = F(g, t)$  for any  $k, k' \in K$ ,  $h \in H$ , and

$$F(g, e) = (R^* v)(gK) = \int_K w(gk) dk.$$

Let  $P \in \mathbf{D}(X)$  be given. Since  $K$  is compact the coset space  $X = G/K$  is reductive and there exists  $D \in \mathbf{D}(G)^K$  such that ([9], p. 285)

$$(28) \quad (Pf)(gK) = D_{(g)}(f(gK))$$

for  $f \in C^\infty(X)$ ,  $g \in G$ .

To transfer derivatives from  $g$  to  $t$  we observe that, by the invariance of  $D$  under left translation by  $gk$  and right translation by  $k$ ,

$$D_{(g)}w(gkt) = D_{(x)}w(gkxt)|_{x=e},$$

where  $g, x, t$  are variables in  $G$ . Integrating over  $K$  it follows that

$$(29) \quad D_{(g)}F(g, t) = D_{(x)}F(g, xt)|_{x=e},$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators

$$K_1^{\beta_1} \cdots K_p^{\beta_p} T_1^{\alpha_1} \cdots T_r^{\alpha_r} H_1^{\gamma_1} \cdots H_q^{\gamma_q}$$

(where all exponents are positive integers) are a basis of  $\mathbf{D}(G)$ . Setting apart the terms with  $\beta = \gamma = 0$ , we can thus write, for some  $E_i, F_j \in \mathbf{D}(G)$  and some constant coefficients  $a_\alpha$ ,

$$(30) \quad D = D' + \sum_{i=1}^p K_i E_i + \sum_{j=1}^q F_j H_j, \quad D' = \sum_{\alpha} a_{\alpha_1 \dots \alpha_r} T_1^{\alpha_1} \cdots T_r^{\alpha_r}.$$

If we replace  $D_{(x)}$  by (30) in (29), the second term  $(K_i E_i)_{(x)} F(g, xt)|_{x=e}$  vanishes because  $K_i \in \mathfrak{k}$  and  $F(g, kxt) = F(g, xt)$ . In the third term the left invariant vector field  $H_j \in \mathfrak{h}$  acts by

$$(H_j)_{(x)} F(g, xt) = \partial_s F(g, x \exp(sH_j) t)|_{s=0},$$

and this vanishes too whenever  $t$  normalizes  $H$ , because  $F(g, xth) = F(g, xt)$ .

Since  $t = e$  in case (i), or  $t \in T$  with  $Ht = tH$  in case (ii), we finally obtain for both cases (in multi-index notation)

$$\begin{aligned} (31) \quad D_{(g)}F(g, t) &= D'_{(x)}F(g, xt)|_{x=e} \\ &= \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} F(g, (\exp s_1 T_1 \cdots \exp s_r T_r) t)|_{s=0} \\ &= \left( \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} \right) F(g, t(s)t)|_{s=0}. \end{aligned}$$

Let the operator  $Q$  be defined by

$$Qf(t) = \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} f(t(s)t)|_{s=0},$$

a right invariant differential operator on the group  $T$  in case (ii). The theorem now follows from (27), (28) and (31) in both cases (i) and (ii).  $\square$