6.6 Examples

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **05.06.2024**

6.6 Examples

Keeping the notations of the previous section, we shall illustrate Theorem 17.

a. Totally geodesic transform. As in Section 4.1 **a**, let $X = G/K$ be a Riemannian symmetric space of the noncompact type and $y_o = \operatorname{Exp}\mathfrak{s}$ the origin in the dual space $Y = G/H$. By (3) we have $\mathfrak{k}+\mathfrak{h} = \mathfrak{k}\oplus\mathfrak{s}$, therefore Theorem 17 (i) applies with $\mathfrak{t} = \mathfrak{s}^{\perp}$, the orthogonal of $\mathfrak{s}$ in $\mathfrak{p}$.

b. Horocycle transform. Again $X = G/K$ is a Riemannian symmetric space of the noncompact type (see Notations, **d**), but the dual space is now the space of horocycles $Y = G/MN$. We recall Harish-Chandra's isomorphism of algebras ([9], p. 306)

$$\Gamma : \mathbf{D}(X) \longrightarrow \mathbf{D}(A)^W ,$$

where $\mathbf{D}(A)^W$ is the subalgebra of W-invariant differential operators in $\mathbf{D}(A)$. The definition of Γ will be recalled during the next proof.

Proposition 18. *Given $v \in C^\infty(Y)$, the function of $x = gK$ and $a \in A$ given by*

$$w(x,a) = a^\rho R_a^* v(x) = a^\rho \int_K v(gkaN)\, dk$$

is a solution of the system of multitemporal wave equations

$$P_{(x)} w(x,a) = \Gamma(P)_{(a)} w(x,a)\,, \quad P \in \mathbf{D}(X), x \in X, a \in A\,.$$

Proof. Theorem 17 (ii) applies here with $T = A$, the abelian subgroup from the Iwasawa decomposition $G = KAN$; indeed $\mathfrak{k}+\mathfrak{h} = \mathfrak{k}+\mathfrak{m}+\mathfrak{n} = \mathfrak{k}\oplus\mathfrak{n}$, and $\mathfrak{g} = (\mathfrak{k}\oplus\mathfrak{n})\oplus\mathfrak{a}$, $[\mathfrak{a},\mathfrak{h}] \subset [\mathfrak{a},\mathfrak{m}] + [\mathfrak{a},\mathfrak{n}] \subset \mathfrak{n} \subset \mathfrak{h}$. By (31) we thus have

$$P_{(x)} R_a^* v(x) = D'_{(a)} R_a^* v(x)\,, \tag{32}$$

where $D \in \mathbf{D}(G)^K$ is related to P by (28) and $D' \in \mathbf{D}(A)$ was characterized by

$$D - D' \in \mathfrak{k}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{n}\,. \tag{33}$$

To compare D' and $\Gamma(P)$ we recall that $\Gamma(P) = a^{-\rho} D_\mathfrak{a} \circ a^\rho$, where $D_\mathfrak{a} \in \mathbf{D}(A)$ is characterized by

$$D - D_\mathfrak{a} \in \mathfrak{n}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{k}\,. \tag{34}$$

Moreover $(Df)(a) = D_\mathfrak{a}(f(a))$ for $a \in A$, if $f \in C^\infty(G)$ is such that $f(ngk) = f(g)$ for any $g \in G$, $k \in K$, $n \in N$ ([9], p. 302 sq.).

Taking $u \in \mathcal{D}(G)$ we have, by a classical integral formula,

$$\int_G Df(g) \cdot u(g)\, dg = \int_{N\times A\times K} Df(a) \cdot u(nak)\, a^{-2\rho} dn\, da\, dk \tag{35}$$
$$= \int_{N\times A\times K} D_{\mathfrak{a}} f(a) \cdot u(nak)\, a^{-2\rho} dn\, da\, dk\,.$$

On the other hand, this integral can be written with the transpose operator ${}^tD$ as

$$\int_G Df(g) \cdot u(g)\, dg = \int_G f(g)\, {}^tDu(g)\, dg$$
$$= \int_A f(a)\, a^{-2\rho} da \int_{N\times K} \left({}^tDu\right)(nak)\, dn\, dk\,.$$

But ${}^tD \in \mathbf{D}(G)^K$ therefore, for any $g \in G$,

$$\int_{N\times K} \left({}^tDu\right)(ngk)\, dn\, dk = \left({}^tD\right)_{(g)} \left(\int_{N\times K} u(ngk)\, dn\, dk\right)\,.$$

The latter integral, as a function of g, is left N-invariant and right K-invariant so that

$$\int_{N\times K} \left({}^tDu\right)(nak)\, dn\, dk = \left({}^tD\right)_{\mathfrak{a}} \left(\int_{N\times K} u(nak)\, dn\, dk\right)\,.$$

Since $\left({}^tD\right)_{\mathfrak{a}} = {}^t\left(D'\right)$ obviously by (33) and (34), we obtain

$$\int_G Df(g) \cdot u(g)\, dg = \int_A D'(f(a)a^{-2\rho})\, da \int_{N\times K} u(nak)\, dn\, dk$$
$$= \int_{N\times A\times K} \left(a^{2\rho}D' \circ a^{-2\rho}\right) f(a) \cdot u(nak)\, a^{-2\rho} dn\, da\, dk\,,$$

for any $f \in C^\infty(A)$ and any $u \in \mathcal{D}(G)$. Comparing with (35) it follows that

$$D_{\mathfrak{a}} = a^{2\rho}D' \circ a^{-2\rho}\,, \quad D' = a^{-\rho}\Gamma(P) \circ a^{\rho}\,,$$

whence the result by (32). $\square$

A slightly different proof can be obtained by decomposing the wave $a^\rho R_a^* v(gK)$ into *elementary horocycle waves* as follows. For $g \in G$ we denote by $A(g) \in A$ the A-component of g in the Iwasawa decompositions $G = NAK = ANK$ (we recall that A normalizes N), and by $K(g) \in K$ its K-component in the decompositions $G = KAN = KNA$.

PROPOSITION 19. (i) *Given* $f \in C^\infty(A)$ *and* $k \in K$, *the function*

$$w(gK, a) = a^{-\rho} f(A(k^{-1}g)a)$$

is a solution of the system of multitemporal wave equations

$$P_{(x)} w(x, a) = \Gamma(P)_{(a)} w(x, a), \quad P \in \mathbf{D}(X),\ x \in X,\ a \in A.$$

(ii) *Given* $v \in C^\infty(Y)$, *the function of* $x = gK$ *and* $a \in A$ *given by*

$$a^\rho R_a^* v(gK) = \int_K a^\rho v(gkaN)\, dk$$

is a solution of the same equations.

REMARKS. Part (i) is Proposition 8.5 in [12], p. 118. Note that, k being fixed, the "wave surfaces" $A(k^{-1}g) = constant$ are parallel horocycles with the same normal $kM \in K/M$ (cf. [11], p. 81). Indeed the equality $A(k^{-1}g) = a_o \in A$ is equivalent to $k^{-1}g \in a_o NK$, i.e. $g \cdot x_o \in ka_o \cdot y_o$.

If λ is a linear form on $\mathfrak{a}$ and $f(a) = a^{i\lambda+\rho}$, the result (i) implies that $A(k^{-1}g)^{i\lambda+\rho}$ is, as a function of gK, an eigenfunction of all invariant operators $P \in \mathbf{D}(X)$; this is a fundamental result for harmonic analysis on X.

Part (ii) provides a simpler proof and a generalization of Proposition 8.6 in [12], p. 118, where v was the Radon transform Ru of some $u \in \mathcal{D}(X)$. We refer to [12] or [13] for a detailed study of those multitemporal wave equations.

Proof of Proposition 19. (i) Both sides of the wave equation are invariant under the action of K on X; we can therefore assume $k = e$. Now $w(gK, a) = a^{-\rho} f(A(g)a)$ is left N-invariant and right K-invariant as a function of g, and it will suffice to prove the result for $g = a \in A$.

By the decomposition (34) of D we have, for any $b \in A$,

$$D_{(g)} \left(f(A(g)b)\right)\big|_{g=a} = (D_{\mathfrak{a}})_{(a)} (f(ab)) = a^\rho \Gamma(P)_{(a)} \left(a^{-\rho} f(ab)\right).$$

But $\Gamma(P)$ is an invariant differential operator on A, isomorphic to the additive group of a vector space, and we obtain

$$\begin{aligned} D_{(g)} \left(b^{-\rho} f(A(g)b)\right)\big|_{g=a} &= a^\rho \Gamma(P)_{(a)} \left((ab)^{-\rho} f(ab)\right) \\ &= a^\rho \Gamma(P)_{(b)} \left((ab)^{-\rho} f(ab)\right) \\ &= \Gamma(P)_{(b)} \left(b^{-\rho} f(ab)\right) = \Gamma(P)_{(b)} \left(b^{-\rho} f(A(g)b)\right)\big|_{g=a}. \end{aligned}$$

Thus (i) is proved for $g = a$.

(ii) Let $g \in G$, $k \in K$ and $k' = K(gk)$. Then $gk = k'a'n'$ with $a' \in A$ and $n' \in N$. It follows that $k'^{-1}g = a'n'k^{-1}$, therefore $a' = A(k'^{-1}g)$ and

$$gkaN = k'A\left(k'^{-1}g\right)aN.$$

For fixed g the map $k \mapsto K(gk) = k'$ is a diffeomorphism of K onto itself and, by the integral formula ([9], p. 197)

$$\int_K F(k')\,dk = \int_K A(k'^{-1}g)^{2\rho}\,F(k')\,dk',$$

we have

$$\begin{aligned} a^\rho R_a^* v(gK) &= a^\rho \int_K v(gkaN)\,dk \\ &= a^\rho \int_K v(k'A(k'^{-1}g)aN)\,dk \\ &= a^{-\rho} \int_K \left(A(k'^{-1}g)a\right)^{2\rho}\, v(k'A(k'^{-1}g)aN)\,dk'. \end{aligned}$$

By (i) applied to the functions $f(a) = a^{2\rho}v(k'aN)$, $k' \in K$, this is a solution of the wave equations. □

COROLLARY 20 (Helgason). *If $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras, there exists a differential operator $P \in \mathbf{D}(X)$ such that the horocycle Radon transform of $X = G/K$ is inverted by*

$$u(x) = PR^*Ru(x)$$

for $u \in \mathcal{D}(X)$, $x \in X$.

We prove it here by means of shifted transforms and wave equations; see [11], p. 116 for Helgason's original proof.

Proof. The assumption on $\mathfrak{g}$ implies that, in the notation of (15), $C \cdot |c(\lambda)|^{-2}$ is a W-invariant polynomial on $\mathfrak{a}^*$. Let $P \in \mathbf{D}(X)$ be the corresponding operator under the isomorphism $\Gamma\colon \mathbf{D}(X) \to \mathbf{D}(A)^W$, so that $\Gamma(P)(i\lambda) = C \cdot |c(\lambda)|^{-2}$. By Theorem 13 and Proposition 19 (ii) (with $v = Ru$)

we have

$$\begin{aligned} u(x) &= \langle T_{(a)}, a^{\rho} R_a^* Ru(x) \rangle \ = \Gamma(D)_{(a)} \left(a^{\rho} R_a^* Ru(x) \right) \big|_{a=e} \\ &= P_{(x)} \left(a^{\rho} R_a^* Ru(x) \right) \big|_{a=e} = P_{(x)} R^* Ru(x) \,. \qquad \square \end{aligned}$$

REFERENCES

[1] BERENSTEIN, C. and E. C. TARABUSI. Inversion formulas for the k-dimensional Radon transform in real hyperbolic spaces. *Duke Math. J. 62* (1991), 613–631.

[2] —— An inversion formula for the horocyclic Radon transform on the real hyperbolic space. *Lectures in Appl. Math. 30* (1994), 1–6.

[3] ERDELYI, A., W. MAGNUS, F. OBERHETTINGER and F. TRICOMI. *Higher Transcendental Functions. Vol. I.* McGraw-Hill, 1953.

[4] GEL'FAND, I., M. GRAEV and Z. SHAPIRO. Differential forms and integral geometry. *Funct. Anal. Appl. 3* (1969), 101–114.

[5] GRINBERG, E. Spherical harmonics and integral geometry on projective spaces. *Trans. Amer. Math. Soc. 279* (1983), 187–203.

[6] GUILLEMIN, V. and S. STERNBERG. *Geometric Asymptotics.* Math. Surveys and Monographs no. 14. Amer. Math. Soc., 1990.

[7] HELGASON, S. The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds. *Acta Math. 113* (1965), 153–180.

[8] —— *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, 1978.

[9] —— *Groups and Geometric Analysis.* Academic Press, 1984.

[10] —— The totally-geodesic Radon transform on constant curvature spaces. In: *Integral Geometry and Tomography. Contemp. Math. 113* (1990), 141–149.

[11] —— *Geometric Analysis on Symmetric Spaces.* Math. Surveys and Monographs no. 39. Amer. Math. Soc., 1994.

[12] —— Radon transforms and wave equations. *Lecture Notes in Math. 1684* (1998), 99–121.

[13] —— Integral geometry and multitemporal wave equations. *Comm. Pure Appl. Math. 51* (1998), 1035–1071.

[14] —— *The Radon Transform.* 2nd edition, Birkhäuser, 1999.

[15] KOBAYASHI, S. and K. NOMIZU. *Foundations of Differential Geometry. Vol. II.* Wiley, 1969.

[16] RADON, J. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. Nat. Kl. 69* (1917), 262–277.