1. Introduction

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VITALI'S CONVERGENCE THEOREM ON TERM BY TERM INTEGRATION

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1. INTRODUCTION

In this article we discuss a convergence theorem of Vitali [29] which appeared in 1907, before Lebesgue proved the dominated convergence theorem. This theorem is in some ways stronger than the standard convergence theorems, and deserves to be better known than it is. Vitali proves that if a sequence of integrable functions f_n converges a.e. to an integrable function f (on a space of finite measure), then the integrals of f_n on any measurable subset converge to those of f, if and only if the integrals are uniformly absolutely continuous. The hard part is to show that convergence of the integrals on any measurable subset implies uniform absolute continuity. Subsequently (in 1915) de la Vallée Poussin [27] simplified Vitali's proof (this is also not well-known), and in 1922 Hahn [10] was led to prove the much better known Vitali-Hahn-Saks Theorem. If Vitali's paper is quoted today, it is usually either as (i) a forerunner to the Vitali-Hahn-Saks Theorem, or (ii) as the much weaker result that L^1 convergence is equivalent to uniform absolute continuity. Note that Vitali's result shows that when f_n converges a.e. to f, weak convergence implies strong convergence. We give here, in modern language and notation, Vitali's original proof, de la Vallée Poussin's simplification, and finally, Hahn's original proof of the Vitali-Hahn-Saks Theorem. This last is also not well-known, having been superseded by the Baire category proof of Saks [25] (and Banach [1]). This article is not directed to experts in the history of the subject, but to the vast majority of real analysts, who though they teach the subject, are not aware of the history or existence of these proofs. Numbers in square brackets refer to the reference list at the end of the article.

The three standard convergence theorems are:

1. THE MONOTONE CONVERGENCE THEOREM (MCT). *If* $\{f_n\}$ *is a sequence of non negative measurable functions on a measurable set* E *with* $f_n \leq f_{n+1}$ *for all* $n \in \mathbf{N}$, *then* $\int_E \lim f_n = \lim \int_E f_n$.

2. FATOU'S LEMMA (Fatou). *If* $\{f_n\}$ *is a sequence of non negative measurable functions on a measurable set* E, *then* $\int_E \liminf f_n \leq \liminf \int_E f_n$.

[The non-negativity in Fatou and the inequalities in MCT may hold a.e.]

3. THE DOMINATED CONVERGENCE THEOREM (DCT). *If* $\{f_n\}$ *is a sequence of measurable functions on a measurable set* E *such that* $f_n \to f$ *a.e. on* E *and if there exists a function* g, *integrable on* E *with* $|f_n| \leq g$ *a.e. on* E, *then* $\int_E f = \lim \int_E f_n$.

A special case is

THE BOUNDED CONVERGENCE THEOREM (BCT): *If* E *is a set of finite measure,* $\{f_n\}$ *measurable on* E *such that* $f_n \to f$ *a.e. on* E, *and if there exists a real number* $M > 0$ *with* $|f_n| \leq M$ *a.e. on* E, *then* $\int_E f = \lim \int_E f_n$.

They are most usually proved in this order; sometimes BCT is proved first. Other convergence results involving mean convergence come somewhat later. DCT is the result most often used in practice (though MCT has perhaps a deeper theoretical significance: see below).

Historically, things were very different. Lebesgue's thesis [14] (referred to as *'thèse'* in what follows) appeared in 1902 as a paper in *Annali di Matematica* entitled « Intégrale, longueur, aire ». Lebesgue's main interests were in various ways of constructing the integral, or primitive, in differentiation and the fundamental theorem of the calculus, and in completing the study of measure, initiated and carried quite far by E. Borel. Convergence theorems were not his main interest, and the thesis contains only BCT on p. 259.

Sketch of his proof. Let $\varepsilon > 0$ and let $F_n = \bigcup_{j \geq n} \{|f_j - f| \geq \varepsilon\}$. Then $m(F_n) \to 0$, since $f_n \to f$ a.e. ($m(F_n)$ denotes the measure of F_n). If $E_n = E \backslash F_n$, then

$$\left|\int_E f_n - \int_E f\right| \le \int_E |f_n - f| = \int_{E_n} |f_n - f| + \int_{F_n} |f_n - f|$$
$$\le \varepsilon m(E) + 2Mm(F_n)$$
$$< \varepsilon m(E) + 2\varepsilon M \text{ for sufficiently large } n.$$

Lebesgue lectured on his new work at the Collège de France in 1902–03, and these lectures were published as a book entitled *« Leçons sur l'intégration et la recherche des fonctions primitives »* (1st edition 1904 [15], referred to as *Leçons I*). Again, the only convergence theorem proved is BCT, on p. 114. But in the last chapter (Chapter VII) of *Leçons I* (Chapter VII also in the revised 2nd edition of 1928 [19], referred to as *Leçons II*), Lebesgue states the « Problème de l'intégration », six properties which an integral on a suitable class of bounded functions should possess. Property (6) is the convergence property: for $f_n, f \ge 0$ and bounded, if $f_n \le f_{n+1}$ for all n, and $f_n \uparrow f$ then $\int f_n \to \int f$. Of course this restricted version of MCT for bounded functions follows at once from BCT. [The best historical account of the theory of integration up to 1910 is in Hawkins [12].]

In 1906 (four years after Lebesgue's thesis was published) Beppo Levi [20] proved MCT and independently Fatou [7] proved his lemma. Levi's paper is short and crystal clear, even if your Italian is rudimentary !

Sketch of his proof. Let $0 \le f_n \le f_{n+1}$ and $f = \lim f_n$ on E. Assume $m(E) < \infty$. Let $f^k = \min(f, k)$, $f_n^k = \min(f_n, k)$, $k \in \mathbf{N}$, and let $a_k = \int_E f^k$, $a_{n,k} = \int_E f_n^k$. Then $a_{n,k}$ is increasing in n for fixed k, and increasing in k for fixed n, so $\lim_n \lim_k a_{n,k} = \lim_k \lim_n a_{n,k}$ regardless of whether the limits are finite or infinite. Since f_n^k, f^k are bounded by k (and of course ≥ 0), BCT gives $\lim_n a_{n,k} = a_k = \int_E f^k$ for each k. If f and so f_n are integrable, then $\lim_k a_k = \int_E f$, $\lim_k a_{n,k} = \int_E f_n$, and so $\int_E f = \int_E \lim f_n = \lim \int_E f_n$, by the equality of the repeated limits. This happens if either repeated limit is finite, in particular if $\lim_n \int_E f_n$ is finite. If not, both repeated limits are infinite and $\lim_n \int_E f_n = +\infty$.

Fatou's proof of his lemma is very similar. It should be noted that Fatou's long paper is one of the most important of the century. For the first time the new theory of integration is applied to complex function theory; there are also fundamental applications to trigonometric series.

It is not until 1908, that DCT first appears in Lebesgue (1908) p. 9–10 [16] with a sketch of the proof; the same thing happens in Lebesgue (1909) [17] at the top of p. 50. In these papers Lebesgue seeks to apply his new results and finds BCT insufficient. In Lebesgue (1910) [18], in § 15 on page 375, the proof of DCT is given in more detail, still on a set of finite measure.

Sketch of his proof. Let $\varepsilon > 0$; since g is integrable on E, there exists a number $M > 0$, such that $\int_F g < \varepsilon$, where $F = \{g > M\}$; then $\int_F |f_n - f| < 2\varepsilon$, and on $E \backslash F$, the result follows by BCT.

Note that all the theorems so far have been stated and proved for sets E of *finite* measure. There does not seem at that time to have been much interest on anyone's part in extending the results and proofs for the case $m(E) = +\infty$. However, (excluding of course BCT) this is easily done.

2. VITALI'S CONVERGENCE THEOREM

In 1907, *before* Lebesgue announced DCT, there appeared a remarkable paper by G. Vitali [29], which, I feel, has not received its due, even from Hawkins. In it Vitali proves the following result:

Let E be a set of finite measure (finiteness is essential here). Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f finite a.e. Then f is integrable and $\int_F f_n \to \int_F f$, for every measurable subset F of E, if and only if the integrals $\int_A f_n$ are uniformly absolutely continuous (uniformly in n): given $\varepsilon > 0$, there exists $\delta > 0$, such that if $m(A) < \delta$, then $\left|\int_A f_n\right| < \varepsilon$ for all n.

This implies that $\int_A |f_n| < 2\varepsilon$. Vitali calls this *equi-absolutely continuous.* Note that this result generalizes at once to any finite measure space.