1. Introduction

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FINITE TYPE LINK-HOMOTOPY INVARIANTS

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ABSTRACT. An explicit polynomial in the linking numbers l_{ij} and Milnor's triple linking numbers $\mu(rst)$ on six component links is shown to be a well-defined finite type link-homotopy invariant. This solves a problem raised by B. Mellor and D. Thurston. An extension of our construction also produces a finite type link invariant which detects the invertibility for some links.

1. INTRODUCTION

The classification of links in 3-space up to link-homotopy [3] was published ten years ago. Since then, the question of whether one could extract link-homotopy invariants from this classification has not been addressed properly. Recall that this classification starts with the classification of k component string links up to link-homotopy by a finitely generated torsion free nilpotent group $\mathcal{H}(k)$. Then link-homotopy classes are classified as orbits of this group $\mathcal{H}(k)$ under the "nilpotent action" of conjugations and partial conjugations. The group $\mathcal{H}(k)$ is of rank

$$\sum_{n=2}^{k}(n-2)!\binom{k}{n},$$

so an element of $\mathcal{H}(k)$ can be described uniquely by that many integers.

These integers are Milnor's μ-numbers [1]) with distinct indices. By a *link-homotopy invariant polynomial*, or simply a link-homotopy invariant, we mean a polynomial in these μ-numbers which is invariant under the action of

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[1]) Usually, they are called μ-invariants. But the word "invariant" is clearly abused here, so we decide to call them μ-numbers.

conjugations and partial conjugations. There are trivial examples of such link-homotopy invariant polynomials coming from polynomials of linking numbers. A link-homotopy invariant polynomial is non-trivial if it contains higher order μ-numbers.

The main result of this paper is that such a non-trivial link-homotopy invariant polynomial exists when $k \geq 6$.

The abelianization of $\mathcal{H}(k)$ is a free abelian group of rank $\binom{k}{2}$. This is where the classical linking numbers l_{ij}, $1 \leq i < j \leq k$, fit in. The action of conjugations and partial conjugations on this quotient of $\mathcal{H}(k)$ is trivial. The next successive quotient of the lower central series of $\mathcal{H}(k)$ is a free abelian group of rank $\binom{k}{3}$, whose elements can be described by the collection of Milnor's triple linking numbers $\{\mu(rst)\,;\ 1 \leq r < s < t \leq k\}$. The conjugations and partial conjugations act on this quotient by translations whose translation vectors' coordinates are linear functions of the linking numbers l_{ij}. Thus, if the dimension of the subspace generated by these translation vectors is less than $\binom{k}{3}$ for generic values of the linking numbers, we may find a non-trivial vector perpendicular to all these translation vectors. Furthermore, the coordinates of this vector could be taken as polynomials in l_{ij}. Then the projection of a vector $\{\mu(rst)\}$ to this perpendicular vector will be invariant under conjugations and partial conjugations. This is the general philosophy behind our construction of link-homotopy invariant polynomials.

The theory of finite type invariants is a general framework for the study of invariants of knots and links. See [1] for an introduction to this theory. Intuitively speaking, multiple crossing switchings on links in 3-space give rise to a very natural filtration on the set of all links and a link invariant is said to be of finite type if it vanishes on all sufficiently deep strata of this filtration.

In a recent preprint [9], B. Mellor and D. Thurston have established the existence of link-homotopy invariants of finite type which are not polynomials of linking number when $k \geq 9$. Their proof is not constructive and therefore it is not clear whether their link-homotopy invariants are polynomials of μ-numbers.

On the other hand, since μ-numbers are of finite type for string links ([7], [2]), it is easy to see that our link-homotopy invariant polynomials are of finite type for links. For $k \leq 5$, it is shown in [9] that the only finite type link-homotopy invariants are polynomials in the linking numbers. So our construction fits nicely with this work of Mellor and Thurston.

Recall that the only finite type knot concordance invariant is the Arf invariant [10]. Since link concordance implies link-homotopy, our work (as well as the work of Mellor and Thurston, of course) shows the existence of non-trivial finite type link concordance invariants.

To extend the applicability of our general philosophy slightly, we find that the operation on the vector $\{\mu(rst)\}$ induced by reversing the orientation of each component of a string link is to change it by a negative sign followed by a translation whose translation vector's coordinates are quadratic polynomials in l_{ij}. If the dimension of the subspace generated by this vector together with the translation vectors of conjugations and partial conjugations is still less than $\binom{k}{3}$ for generic values of the linking numbers, and this is the case indeed, we can construct a non-trivial link-homotopy invariant polynomial which is changed by a sign when the orientation of each component of a link is reversed. We say that such a link invariant detects the invertibility for links. Recall that the reversion of the orientation of every component of a link does not change the quantum invariant associated with an irreducible representation of a semi-simple Lie algebra (see, for example, [8]). Thus our invariant is of finite type but is not determined by quantum invariants. The existence of a finite type knot invariant which detects the invertibility for knots is a major problem in the theory of finite type invariants (see, for example, [8] and [4]). We believe that finite type knot invariants can not detect the invertibility for knots.

It remains unclear whether we can have a complete set of link-homotopy invariant polynomials which determines uniquely link-homotopy classes of links. See [5] for an earlier attempt on this problem[2]). This problem could probably be translated to the problem of understanding the sublattice generated by the translation vectors of conjugations and partial conjugations. A better understanding of this sublattice might also be useful in answering the following question. If we let $\deg(l_{ij}) = 1$ and $\deg(\mu(rst)) = 2$, the link-homotopy invariant polynomial for $k = 6$ we construct in Section 3, which detects the invertibility for links, is a linear combination of 113,700 monomials of degree 22, homogeneous in both l_{ij} and $\mu(rst)$ and linear in $\mu(rst)$. Is there a shorter link-homotopy invariant polynomial detecting the invertibility for links ?

2. Conjugation and partial conjugation

We first recall the classification of ordered, oriented links up to link-homotopy given in [3]. We will follow the notations of [3].

[2]) See [6] for another approach to the similar problem for surgery equivalence of links. Notice that both approaches attempted to reduce the indeterminacies of the $\bar{\mu}$-invariants.