4. The group of homeomorphisms of the circle

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4. THE GROUP OF HOMEOMORPHISMS OF THE CIRCLE

We denote by $\mathrm{Homeo}_+(\mathbf{S}^1)$ the group of orientation preserving homeomorphisms of the circle $\mathbf{S}^1$. In this section, we want to describe the main properties of this group.

Denote by $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ the group of homeomorphisms $\widetilde{f}$ of the real line $\mathbf{R}$ which satisfy $\widetilde{f}(x+1)=\widetilde{f}(x)+1$ for all x, *i.e.* which commute with integral translations. Every element of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ defines a homeomorphism of the circle which is orientation preserving, so that we get a homomorphism p from $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. The kernel of p consists of integral translations of the real line. Moreover p is onto: any orientation preserving homeomorphism of the circle lifts to a homeomorphism of its universal covering space, which is the line $\mathbf{R}$, commuting with integral translations. In other words, we have an exact sequence:

$$0 \to \mathbf{Z} \to \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \to \mathrm{Homeo}_+(\mathbf{S}^1) \to 1\,.$$

We say that $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ is a central extension of $\mathrm{Homeo}_+(\mathbf{S}^1)$.

Equipped with the topology of uniform convergence, these groups are naturally topological groups.

We would like to turn these groups into infinite dimensional Lie groups. It is not so easy to do so for many reasons. One of the difficulties is that it is not true that an element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ close to the identity lies on a 1-parameter subgroup (see 5.10). For an excellent survey on infinite dimensional Lie groups, we refer to [53]. In any case, it is customary to think of these homeomorphism groups as "some kind of infinite dimensional Lie groups". For a recent study of the "Lie algebra" of $\mathrm{Homeo}_+(\mathbf{S}^1)$, see [47].

Even though $\mathrm{Homeo}_+(\mathbf{S}^1)$ is not quite a Lie group, it shares many properties with finite dimensional Lie groups. More precisely, we shall try to show that $\mathrm{Homeo}_+(\mathbf{S}^1)$ is a kind of infinite dimensional analogue of $\mathrm{PSL}(2,\mathbf{R})$.

Lie groups admit a maximal compact subgroup K, unique up to conjugacy, and the embedding of K in the Lie group is a homotopy equivalence (see for instance [58]). In case of $\mathrm{PSL}(2,\mathbf{R})$, the maximal compact subgroup is $\mathrm{SO}(2,\mathbf{R})/\{\pm \mathrm{Id}\} \simeq \mathbf{R}/\mathbf{Z}$ and the quotient of $\mathrm{PSL}(2,\mathbf{R})$ by its maximal compact subgroup is contractible since it is identified with the upper half space $\mathcal{H}$ (we remark that $\mathrm{PSL}(2,\mathbf{R})$ acts transitively on $\mathcal{H}$ and that the stabilizer of a point is a maximal compact subgroup).

The same result is true for the group of homeomorphisms of the circle:

PROPOSITION 4.1. *Up to conjugacy, the rotation group* $\mathrm{SO}(2,\mathbf{R})$ *is the only maximal compact subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$.

Proof. Let K be a compact subgroup in $\mathrm{Homeo}_+(\mathbf{S}^1)$ and λ its Haar probability measure, *i.e.* the unique probability measure on K which is invariant under left (and right) translations in K (see for instance [69]). Each element k of K sends the Lebesgue measure dx of the circle $\mathbf{R}/\mathbf{Z}$ to a probability measure $k_\star dx$ on the circle. Averaging using λ, we get a probability measure $\mu = \int_K (k_\star dx)\, d\lambda$ on the circle which is invariant under the action of K. This measure μ obviously has no atom and is non zero on non empty open sets. It follows that there is an orientation preserving homeomorphism h of the circle such that $h_\star\mu = dx$. This is a very special case of a theorem which is valid in any dimension but the proof is very easy on the circle. Indeed, fix a point x_0 on the circle (for instance $0 \mod \mathbf{Z}$) and define $h(x)$ to be the unique point such that the μ-measure of the positive interval from x_0 to x is equal to the Lebesgue measure of the positive interval from x_0 to $h(x)$. The existence and continuity of h follow from the properties of μ and the fact that h sends μ to the Lebesgue measure is obvious from the definition. Now, after conjugating K by h, we get a group of orientation preserving homeomorphisms of the circle preserving the Lebesgue measure, *i.e.* a group of rotations. Hence, some conjugate of K is contained in $\mathrm{SO}(2,\mathbf{R})$. $\square$

Note in particular that the proposition implies that *any finite subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is cyclic and is conjugate to a cyclic group of rotations.*

PROPOSITION 4.2. *The embedding of* $\mathrm{SO}(2,\mathbf{R})$ *in* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is a homotopy equivalence.*

Proof. Observe first that the group of orientation preserving homeomorphisms of the line $\mathbf{R}$ is a convex set since it is the set of strictly increasing functions from $\mathbf{R}$ to $\mathbf{R}$ tending to $\pm\infty$ as the variable tends to $\pm\infty$. Consider the group $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. An element $\widetilde{f}$ of this group can be written in the form $\widetilde{f}(x) = x + t(x)$ where t is a $\mathbf{Z}$-periodic function. Now, any such periodic function can be written in a unique way in the form $c_0 + t_0$ where c_0 is a constant and t_0 is a periodic function whose average value over a period is 0. If $0 \leq s \leq 1$ is a parameter, we define $\widetilde{f}_s$ by $\widetilde{f}_s(x) = x + c_0 + (1-s)t_0(x)$. We have $\widetilde{f}_0 = \widetilde{f}$ and $\widetilde{f}_1$ is a translation. It follows from our preliminary observation that for each s in $[0,1]$, $\widetilde{f}_s$ is a homeomorphism in $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$. Hence we get a continuous retraction of $\widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ on the subgroup of translations of $\mathbf{R}$, isomorphic to $\mathbf{R}$. Note also that this retraction commutes with integral translations, since the average value of $t(x)+1$ over a period is of course c_0+1. In other words, we can define a continuous retraction from the quotient group $\mathrm{Homeo}_+(\mathbf{S}^1) = \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)/\mathbf{Z}$ onto the group of rotations $\mathrm{SO}(2,\mathbf{R}) \simeq \mathbf{R}/\mathbf{Z}$. This is the homotopy equivalence that we were looking for. Observe that we actually proved something a little bit stronger: $\mathrm{Homeo}_+(\mathbf{S}^1)$ is homeomorphic to the product of $\mathrm{SO}(2,\mathbf{R})$ and a convex set. □

We should not only consider $\mathrm{Homeo}_+(\mathbf{S}^1)$ as a kind of Lie group but as an analogue of a *simple* Lie group (as for example $\mathrm{PSL}(2,\mathbf{R})$) for which there is a well developed and wonderful theory (see for instance [58]).

THEOREM 4.3. *The group* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is simple.*

Proof. Recall that if Γ is any group, its *first commutator subgroup* $\Gamma' \subset \Gamma$ is the subgroup generated by commutators $[\gamma_1,\gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ of elements γ_1,γ_2 in Γ (see [46]). A group is called *perfect* if it is equal to its first commutator group, *i.e.* if every element is a product of commutators.

We shall establish later that $\mathrm{Homeo}_+(\mathbf{S}^1)$ is perfect (see 5.11). (Note that the corresponding statement for diffeomorphism groups is also true and much harder to prove but we decided not to discuss groups of diffeomorphisms...) We now show that this implies quite formally the simplicity of $\mathrm{Homeo}_+(\mathbf{S}^1)$. The reader will find in [18] a general theorem stating that a perfect group of homeomorphisms of a manifold which is acting "sufficiently transitively on finite sets" is necessarily a simple group (see also [2]). The proof we present here is an adaptation of this argument.

Recall that the *support* of a homeomorphism is the closure of the set of points which are not fixed. Let N be a non trivial normal subgroup in $\mathrm{Homeo}_+(\mathbf{S}^1)$ and suppose f is some element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ whose support is contained in some compact interval $I \subset \mathbf{S}^1$. Let n_0 be a non trivial element of N and choose some closed interval in $\mathbf{S}^1$ which is disjoint from its image under n_0. Observe that $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts transitively on closed intervals in the circle. Conjugating n_0 by a suitable element of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we can therefore find an element n in N such that $n(I)$ is disjoint from I. Consider now the commutator $g = n^{-1}f^{-1}nf$. It is an element of N since N is a normal subgroup. Moreover, it is clear that g agrees with f on I, with $n^{-1}f^{-1}n$ on $n^{-1}(I)$, and with the identity in the complement of these two disjoint intervals.

Consider now two elements f_1 and f_2 of $\mathrm{Homeo}_+(\mathbf{S}^1)$ whose supports are contained in the same interval I. We can find elements n_1 and n_2 of N such that the intervals I, $n_1^{-1}(I)$ and $n_2^{-1}(I)$ are disjoint. Then the two elements

$g_1 = n_1^{-1}f_1^{-1}n_1f_1$ and $g_2 = n_2^{-1}f_2^{-1}n_2f_2$ are in N and their commutator is equal to the commutator of f_1 and f_2. So we showed that *the commutator of two elements of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *having support contained in the same interval is in* N.

Cover the circle by three intervals I_1, I_2, I_3 with empty triple intersection but with non empty intersection two by two. Let G_1, G_2, G_3 be the subgroups of homeomorphisms with supports in I_1, I_2 and I_3 respectively and denote by G the subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ that they generate. If a group is generated by a subset, its first commutator subgroup is generated by conjugates of commutators of elements in this subset. It follows that the first commutator subgroup of G is generated by conjugates of commutators of elements in $G_1 \cup G_2 \cup G_3$. Since the union of two of the intervals I_1, I_2, I_3 is not the full circle, it is contained in some compact interval. Hence we can use the above argument to conclude that the commutator of two elements in $G_1 \cup G_2 \cup G_3$ is in N. It follows that N *contains the first commutator subgroup of* G.

We finally prove that G coincides with $\mathrm{Homeo}_+(\mathbf{S}^1)$ and this will prove the proposition since we know that $\mathrm{Homeo}_+(\mathbf{S}^1)$ is equal to its first commutator subgroup (actually we postponed the proof to 5.11 !). Let $x_{1,2}, x_{2,3}, x_{3,1}$ be points in the interiors of $I_1 \cap I_2, I_2 \cap I_3, I_3 \cap I_1$ respectively. Let f be an element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ close enough to the identity so that $f(x_{1,2}), f(x_{2,3}), f(x_{3,1})$ are in the interiors of $I_1 \cap I_2, I_2 \cap I_3, I_3 \cap I_1$ respectively. Then, we can find (commuting) elements g_1, g_2, g_3 of $\mathrm{Homeo}_+(\mathbf{S}^1)$ with supports in $I_1 \cap I_2, I_2 \cap I_3, I_3 \cap I_1$ respectively, agreeing with f in neighbourhoods of $x_{1,2}, x_{2,3}, x_{3,1}$. Hence $g_1^{-1}g_2^{-1}g_3^{-1}f$ is the identity in neighbourhoods of $x_{1,2}, x_{2,3}, x_{3,1}$ and is therefore a product of three elements of $G_1 \cup G_2 \cup G_3$. This shows that every element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ close enough to the identity is an element of G. The general case follows from the well known fact that a connected topological group is generated by any neighbourhood of the identity. □

As a corollary of Proposition 4.2, the fundamental group of $\mathrm{Homeo}_+(\mathbf{S}^1)$ is $\mathbf{Z}$ so that for each integer $k \geq 1$ there is a unique connected covering space $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ of $\mathrm{Homeo}_+(\mathbf{S}^1)$ with k sheets. In the same way, there is a unique connected covering space $\mathrm{PSL}_k(2, \mathbf{R})$ of $\mathrm{PSL}(2, \mathbf{R})$ with k sheets. It is easy to construct these coverings explicitly. Consider a k-fold cover of the circle onto itself. Any element of $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be lifted to exactly k homeomorphisms of the circle and $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ consists of the collection of all the lifts of all homeomorphisms. Another way of expressing the same thing is the following. Consider the finite cyclic group of rotations of order k acting on the circle $\mathbf{R}/\mathbf{Z}$. Then we can consider the subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$

consisting of homeomorphisms commuting with this cyclic group: this is a group isomorphic to $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$. This presentation has the advantage of expressing $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ as a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Analogously, we can realize $\mathrm{PSL}_k(2,\mathbf{R})$ as the group of lifts of elements of $\mathrm{PSL}(2,\mathbf{R})$ to the k-fold cover of $\mathbf{RP}^1$. This k-fold cover is homeomorphic to a circle so that $\mathrm{PSL}_k(2,\mathbf{R})$ can be realized as a subgroup of $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ (of course up to conjugacy). Note however that $\mathrm{PSL}_k(2,\mathbf{R})$ cannot be realized as a subgroup of $\mathrm{PSL}(2,\mathbf{R})$.

Summing up, for each integer $k \geq 1$, we have well defined conjugacy classes of subgroups $\mathrm{PSL}_k(2,\mathbf{R})$ and $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ in $\mathrm{Homeo}_+(\mathbf{S}^1)$. The first ones are finite dimensional and the second ones are very close to the full group of homeomorphisms, something like "finite codimension subgroups".

PROBLEM 4.4. *Let Γ be a closed subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *acting transitively on the circle. Is Γ conjugate to one of the subgroups* $\mathrm{SO}(2,\mathbf{R})$, $\mathrm{PSL}_k(2,\mathbf{R})$ *or* $\mathrm{Homeo}_{k,+}(\mathbf{S}^1)$ *?*

Informally, this problem asks whether there is an interesting geometry on the circle besides projective geometry. For instance, the analogous question for the group of homeomorphisms of the 2-sphere would have a negative answer: besides finite dimensional Lie groups acting on the 2-sphere, there is the group of area preserving homeomorphisms which acts transitively and is "much smaller" than the full group of homeomorphisms of the 2-sphere.

Let F_2 be the free group on two generators. It is very easy to construct explicit examples of embeddings of F_2 in $\mathrm{SL}(2,\mathbf{R})$ (see for instance [31]). It follows that for a generic choice of two elements of $\mathrm{SL}(2,\mathbf{R})$, the subgroup that they generate is free. Indeed, let f and g be two elements in $\mathrm{SL}(2,\mathbf{R})$. There is a homomorphism i from F_2 to $\mathrm{SL}(2,\mathbf{R})$ sending the first and the second generator to f and g respectively. In practice, if w is a non trivial element of F_2 seen as a word in the two generators and their inverses, $i(w) = w(f,g)$ is obtained by substituting f and g to the two generators of F_2 in w. Let $X_w \subset \mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ be the set of (f,g) such that $w(f,g) = \mathrm{Id}$. This is a real algebraic subset of the algebraic irreducible affine variety $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ which is not everything since otherwise $\mathrm{SL}(2,\mathbf{R})$ would not contain any free subgroup on two generators. Therefore the set of couples (f,g) which generate a free subgroup is the complement of a countable union of proper algebraic submanifolds of $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$. Hence for a generic choice of (f,g) (in the sense of Baire), the group generated by (f,g) is free.

We now prove the analogous statement for the group of homeomorphisms (which is not an algebraic group).

PROPOSITION 4.5. *For a generic set of pairs (f,g) of elements of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *(in the sense of Baire), the group generated by (f,g) is a free group on two generators.*

Proof. First observe that the topology of $\mathrm{Homeo}_+(\mathbf{S}^1)$ can be defined by a complete metric. It follows that $\mathrm{Homeo}_+(\mathbf{S}^1)\times\mathrm{Homeo}_+(\mathbf{S}^1)\times\mathbf{S}^1$ is a Baire space, *i.e.* a countable intersection of dense open sets is dense. For each non trivial $w\in F_2$, consider the closed set $X_w\subset\mathrm{Homeo}_+(\mathbf{S}^1)\times\mathrm{Homeo}_+(\mathbf{S}^1)\times\mathbf{S}^1$ consisting of those (f,g,x) such that $w(f,g)(x)=x$. We shall show that all these closed sets have empty interior. It will follow in particular that for each non trivial word w, the set of (f,g) such that $w(f,g)=id$ has empty interior so that, by Baire's theorem, for a generic choice of (f,g) in $\mathrm{Homeo}_+(\mathbf{S}^1)\times\mathrm{Homeo}_+(\mathbf{S}^1)$ there is no non trivial relation of the form $w(f,g)=id$ and the group generated by f and g is indeed free.

Assume by contradiction that some X_w has non empty interior and let w be a word of minimal length k for which this is the case (note that $k>1$). Let U be some non empty open set of $\mathrm{Homeo}_+(\mathbf{S}^1)\times\mathrm{Homeo}_+(\mathbf{S}^1)\times\mathbf{S}^1$ contained in X_w. For each pair of words w_1,w_2 of length strictly less than k, consider the closed set of triples (f,g,x) such that $w_1(f,g)w_2(f,g)(x)=w_2(f,g)(x)$: this is the image of X_{w_1} by $(f,g,x)\mapsto(f,g,w_2^{-1}(f,g)(x))$ and therefore has an empty interior. Choose a triple (f,g,x) which is in U but not in these (finitely many) closed sets with empty interiors. Write $w=a_1.a_2.\cdots.a_k$ where each a_i is one of the generators or its inverse. Write $w(f,g)$ as $f_1f_2\cdots f_k$ where each f_i is one of f,f^{-1},g,g^{-1}.

Finally, consider the sequence of points $x_1,\dots,x_{k-1}$ defined by $x_1=f_1(x)$, $x_2=f_2(x_1)$, $\dots$, $x_{k-1}=f_{k-1}(x_{k-2})$. Since we know that $(f,g,x)\in U$, we have $f_k(x_{k-1})=x$. We claim that the points $x,x_1,\dots,x_{k-1}$ are different. Indeed, the contrary would mean that some word w_1 of length strictly less than k would fix one of the points x_i. Since each point x_i has the form $w_2(f,g)(x)$ for some w_2 of length strictly less than k, the triple (f,g,x) would be in one of these closed sets with empty interior that we excluded.

We slightly modify (f,g) in $(\bar f,\bar g)$ in such a way that the corresponding $\bar f_1,\dots,\bar f_k$ still satisfy $x_1=\bar f_1(x),x_2=\bar f_2(x_1),\dots,x_{k-1}=\bar f_{k-1}(x_{k-2})$ but also such that $\bar f_k(x_{k-1})\neq x$. This is possible since x is different from $x_1,\dots,x_{k-1}$. It follows that $w(\bar f,\bar g)(x)\neq x$. This contradicts the definition of U. $\square$

Note that this proof works equally well for the homeomorphism group of manifolds of any dimension.

Brin and Squier have discovered the remarkable fact that the situation is completely different in groups of piecewise linear homeomorphisms [10].

THEOREM 4.6 (Brin-Squier). *The group* $\mathrm{PL}_+([0,1])$ *of piecewise linear homeomorphisms of the interval* $[0,1]$ *does not contain any non abelian free subgroup.*

Proof. If f is any homeomorphism, we denote by $Supp_0(f)$ its "open support", *i.e.* the set of non fixed points. Suppose by contradiction that there exist two elements f and g of $\mathrm{PL}_+([0,1])$ which generate a free subgroup F_2 on the generators f and g. The union $I = Supp_0(f) \cup Supp_0(g)$ is a union of a finite number of open intervals $I_1, \dots, I_n$. Note that the commutator $fgf^{-1}g^{-1}$ has an open support whose closure is contained in I since near the boundary of I, the maps f and g are linear and therefore commute.

Among the non trivial elements h in F_2 such that the closure of $Supp_0(h)$ is contained in I, consider an element h such that $Supp_0(h)$ meets the least possible number of the n components of I. Let $]a,b[$ be one of these components and let $[c,d]$ be a interval contained in the interior of $]a,b[$ and containing $Supp_0(h) \cap]a,b[$. If x is in $]a,b[$ then the orbit of x under the group F_2 is contained in $]a,b[$ and its upper bound is a common fixed point of f and g so that it has to be b. It follows that there exists an element l in the group sending c (and hence $[c,d]$) to the right of d. In particular the restrictions to $[a,b]$ of h and lhl^{-1} commute and generate a group isomorphic to $\mathbf{Z}^2$. Of course h and lhl^{-1} don't commute in the free group generated by f and g since otherwise they would generate a group isomorphic to $\mathbf{Z}$. Consider now the commutator of h and lhl^{-1}. It is a non trivial element whose support does not intersect $]a,b[$ and therefore intersects strictly fewer components of I than h did. This contradicts our choice of h. □

Finding a group which does not contain any non abelian free subgroups is not very difficult: consider for example an abelian group! However, the interesting feature of $\mathrm{PL}_+([0,1])$ is that it contains no non abelian free subgroups *and satisfies no law.* This means that for every non trivial word w, we can find two elements (f,g) in $\mathrm{PL}_+([0,1])$ such that $w(f,g) \neq \mathrm{Id}$ (this is not difficult: see [10]). Abelian groups, on the contrary, satisfy the law that $fgf^{-1}g^{-1}$ is always the identity element.

Remark also that the proposition is *not* claiming that the group $\mathrm{PL}_+(\mathbf{S}^1)$

does not contain any non abelian free subgroups. Indeed, it is very easy to find free subgroups on two generators in $\mathrm{PL}_+(\mathbf{S}^1)$ using for instance the classical "Klein ping-pong lemma" (see [31] or Section 5.2). Later in this paper, we shall prove that "most subgroups" of $\mathrm{Homeo}_+(\mathbf{S}^1)$ contain free subgroups (5.14).

4.1 Locally compact groups acting on the circle

Recall that a very important (and difficult) theorem of Montgomery and Zippin states that a locally compact group is a Lie group if and only if there is a neighbourhood of the identity which does not contain a non trivial compact subgroup [40, 56]. We know the structure of compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$: they are conjugate to subgroups of $\mathrm{SO}(2,\mathbf{R})$ and therefore they are either finite cyclic groups or conjugate to $\mathrm{SO}(2,\mathbf{R})$. None of these subgroups can be in a small neighbourhood of the identity. Indeed, consider the neighbourhood U of the identity in $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ consisting of those homeomorphisms f such that the distance between x and $f(x)$ is less than $1/3$ for all x in $\mathbf{R}/\mathbf{Z}$. Every element f in U has a unique lift $\widetilde{f}$ in $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$ which is such that $\left|\widetilde{f}(\widetilde{x})-\widetilde{x}\right| \leq 1/3$ for all $\widetilde{x}$ in $\mathbf{R}$. Of course, if f, g and fg are in U, we have $\widetilde{fg} = \widetilde{f}\widetilde{g}$. In particular, if there were a non trivial subgroup H contained in U this subgroup H would lift as a subgroup of $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$. Since $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$ is a torsion free group and since any compact subgroup of $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ contains elements of finite order, it follows that no non trivial compact subgroup of $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ can lift to $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$. In particular U contains no non trivial compact subgroup. We deduce:

Theorem 4.7. *A locally compact subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is a Lie group.*

It would be interesting to prove this theorem by elementary means, *i.e.* without the use of the Montgomery-Zippin theorem.

Consider a *connected Lie group G acting continuously and faithfully* on the circle by a homomorphism $\phi\colon G \to \mathrm{Homeo}_+(\mathbf{S}^1)$. Our objective is to determine all such actions. Orbits of the action are connected, so they can be of three kinds: the full circle, an open interval or a point. In other words, there is a closed set $F \subset \mathbf{S}^1$ (which might be empty) consisting of fixed points for the action, and the orbits which are not fixed points are the connected components of $\mathbf{S}^1 - F$. So, in order to understand the action, it is basically sufficient to understand it on each 1-dimensional orbit (homeomorphic to $\mathbf{R}$ or $\mathbf{S}^1$). Note that the action of G on one orbit is not necessarily faithful anymore but, taking the quotient by the kernel, we are led to study transitive and faithful actions of a connected Lie group G on $\mathbf{R}$ or $\mathbf{S}^1$.

Denote by H the stabilizer of a point in such an orbit. This is a closed subgroup of G, hence a Lie subgroup of codimension 1 and G acts smoothly on the 1-dimensional manifold G/H. The Lie algebra $\mathfrak{G}$ will therefore induce a finite dimensional Lie algebra of smooth vector fields on G/H. Since G acts transitively on G/H, for any point on G/H there is an element of this Lie algebra which does not vanish at this point.

Consider the case of the projective action of $\mathrm{PSL}(2,\mathbf{R})$ on $\mathbf{RP}^1$. The Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ is the algebra of 2×2 real matrices with trace 0. Taking the differential of the action at the identity, one easily checks that the corresponding Lie algebra of vector fields is the algebra of vector fields of the form $u(x)\partial/\partial x$ where u is a polynomial of degree at most 2; thus we get the following identification of algebras:

$$\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbf{R}) \mapsto (b+2ax-cx^2)\frac{\partial}{\partial x}\,.$$

Denote by $\mathfrak{Vect}$ the Lie algebra of germs of smooth vector fields of $\mathbf{R}$ in the neighbourhood of 0. The subspace $\mathfrak{Vect}_k$ of vector fields $u(x)\partial/\partial x$ where u vanishes at the origin together with its first k derivatives is an ideal in $\mathfrak{Vect}$ and the quotient Lie algebra $\mathfrak{Vect}/\mathfrak{Vect}_k$ can be identified, as a vector space with the space $\mathfrak{P}_k$ of vector fields of the form $u(x)\partial/\partial x$ where u is a polynomial of degree at most k.

Note however that $\mathfrak{P}_k$ is a subalgebra of $\mathfrak{Vect}$ if and only if $k=0,1$ or 2. One can therefore think of $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ at the same time as subalgebras of $\mathfrak{Vect}$ and as quotient algebras of $\mathfrak{Vect}$.

The general situation was analyzed a long time ago by Lie, who found all the possibilities [45]:

THEOREM 4.8 (Lie). *Let $\mathfrak{G}$ be a non trivial finite dimensional Lie algebra consisting of germs of smooth vectors fields in the neighbourhood of* 0 *in* $\mathbf{R}$. *Assume that not all these vector fields vanish at the origin. Then the dimension of $\mathfrak{G}$ is at most 3. More precisely, in suitable coordinates $\mathfrak{G}$ consists of all germs of the form $u(x)\partial/\partial x$ where u is a polynomial of degree less than or equal to k for $k=0,1$ or 2.*

Proof. Since one element of $\mathfrak{G}$ does not vanish at the origin, we can find a suitable local coordinate x such that the germ of this element is $\partial/\partial x$. Let $\mathcal{E}$ be the finite dimensional vector space of germs of functions u such that $u(x)\partial/\partial x$ belongs to $\mathfrak{G}$. Of course $\mathcal{E}$ contains the constants and is stable under the operation of taking derivatives, since the bracket $[\partial/\partial x, u(x)\partial/\partial x]$ equals $u'(x)\partial/\partial x$. The successive iterates of the linear operator induced by the derivative must be linearly dependent. This shows that there exists a linear differential equation with constant coefficients which is satisfied by all elements in $\mathcal{E}$. It follows that all elements in $\mathcal{E}$ are real analytic functions. Every non trivial element u of $\mathcal{E}$ therefore has a convergent Taylor expansion of the form $u(x) = a_i x^i + \dots$ with $a_i \neq 0$. Moreover, this integer i is bounded since a solution of a linear differential equation with constant coefficients which vanishes at a point together with its derivatives of orders up to the degree of the equation has to vanish identically. Choose an element u for which the integer i is maximal. Now the algebra $\mathfrak{G}$ contains $[u(x)\partial/\partial x, u'(x)\partial/\partial x] = a_i(x^{2i-2} + \dots)\partial/\partial x$. It follows that $2i-2 \leq i$, so that $i \leq 2$.

For each element of $\mathfrak{G}$, consider the Taylor expansion of degree 2 of the associated vector field, considered as an element of $\mathfrak{P}_2 \simeq \mathfrak{sl}(2,\mathbf{R})$. This produces a linear map $j_2\colon \mathfrak{G} \to \mathfrak{P}_2$ which is clearly an algebra homomorphism and which is injective by the previous argument.

If the image of j_2 is 1-dimensional, then $\mathfrak{G}$ consists only of constant multiples of $\partial/\partial x$. In this case, G is (locally) isomorphic to $\mathbf{R}$ and the Lie algebra of H is trivial, which means that H is discrete.

Suppose that the image of j_2 is 3-dimensional, *i.e.* that j_2 is an isomorphism. Consider the element $X = \partial/\partial x$ of $\mathfrak{P}_2$. Note that the linear operator $ad^3(X)\colon \mathfrak{P}_2 \to \mathfrak{P}_2$ is trivial. The vector field $j_2{}^{-1}(X)$ does not vanish at the origin so that we could have used it at the beginning when we chose a local coordinate x. In other words, there is a local coordinate x such that $\partial/\partial x$ belongs to $\mathfrak{G}$ and such that the linear operator induced by taking bracket with $\partial/\partial x$ is nilpotent of order 3. This means that the third derivative of every element of $\mathcal{E}$ vanishes. In suitable coordinates $\mathfrak{G}$ coincides with polynomial vector fields of degree at most 2. In this case, G is locally isomorphic to $\mathrm{SL}(2,\mathbf{R})$ and H is locally isomorphic to the group of upper triangular matrices.

Suppose finally that the image of j_2 is 2-dimensional. This means that the Taylor expansion of order 1 is an isomorphism $j_1\colon \mathfrak{G} \to \mathfrak{P}_1$ and one can reproduce the above proof with the nilpotent operator of order 2 induced by $\partial/\partial x$. In this case, G is locally isomorphic to the 2-dimensional group of upper triangular matrices in $\mathrm{SL}(2,\mathbf{R})$ and H is locally isomorphic to the 1-dimensional subgroup of unipotent matrices. $\square$

This theorem gives a complete *local* description of transitive actions of a Lie group. It is not difficult to deduce the complete classification of transitive and faithful actions of connected Lie groups on 1-manifolds. Up to conjugacy, the list is the following.

- The action of $\mathbf{R}$ on itself.
- The action of the circle $\mathbf{R}/\lambda\mathbf{Z}$ on itself (for $\lambda > 0$).
- The action of the affine group $\mathrm{Aff}_+(\mathbf{R})$ on $\mathbf{R}$.
- The action of the k-fold cover $\mathrm{PSL}_k(2,\mathbf{R})$ of $\mathrm{PSL}(2,\mathbf{R})$ on the circle, described in Section 4 (for $k \geq 1$).
- The action of the universal cover $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ of $\mathrm{SL}(2,\mathbf{R})$ on the universal cover of $\mathbf{S}^1$.

Loosely speaking, we could say that there are three geometries of finite type on 1-manifolds: euclidean, affine and projective.

The full description of faithful non transitive actions of a connected Lie group G on the circle is now easy in principle. We should choose a closed set $F \subset \mathbf{S}^1$ consisting of fixed points and for each connected component I of the complement of F, the action is described by some surjection from G to $\mathbf{R}$, $\mathrm{Aff}_+(\mathbf{R})$, $\mathrm{PSL}_k(2,\mathbf{R})$ or $\widetilde{\mathrm{SL}}(2,\mathbf{R})$.

As a trivial example, we get the description of *topological flows on the circle*, *i.e.* of actions of $\mathbf{R}$ on the circle. If such a flow is transitive, it is conjugate to the action of $\mathbf{R}$ on the circle $\mathbf{R}/\lambda\mathbf{Z}$ for some $\lambda > 0$ (the "period" of the flow). If it is not transitive, it has a non empty set of fixed points $F \subset \mathbf{S}^1$ and the conjugacy class is completely described by the orientation: on each component of $\mathbf{S}^1 - F$, the flow is positive or negative.

Finally, we should describe the actions of non connected Lie groups G. Let G_0 be the connected component of the identity in G so that we already understand the action of G_0. Observe that G_0 is a normal subgroup of G so that the action of G preserves F and permutes the connected components of $\mathbf{S}^1 - F$. It is not easy to fully analyze this situation but it is quite clear that when G_0 is non trivial, its normalizer is usually very small. We leave to the reader the details of this analysis. Of course, when G_0 is trivial, *i.e.* when G is discrete, the previous discussion has no content. Hence among locally compact groups acting on the circle, the most interesting ones are the discrete groups.