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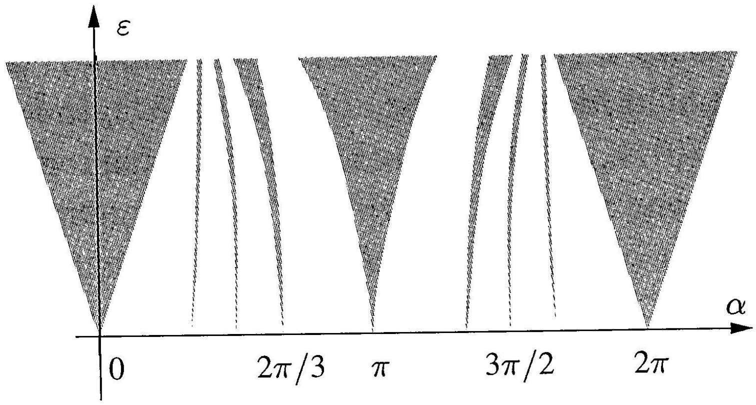

FIGURE 6

As an additional example, consider the case of piecewise linear homeomorphisms of the circle. Since the group $\mathrm{PL}_+(\mathbf{S}^1)$ contains all rotations, it is clear that the rotation number of such a homeomorphism can be arbitrary. However, it is shown in [28] that the rotation number of any element of the Thompson group G is rational and that any rational number is achieved. The proof is very indirect and there is a need for a better proof. We could formulate the problem in the following way.

PROBLEM 5.13. *Consider a rational piecewise linear homeomorphism f of the circle, i.e. such that all its slopes are rational and such that all "break-points" are rational. Is it true that the rotation number of f is rational?*

We can in fact generalize Thompson's group quite a lot in the following way. Let $\Lambda \subset \mathbf{R}_\star^+$ be a subgroup of the multiplicative group of positive real numbers and let $W \subset \mathbf{R}$ be an additive subgroup invariant under multiplication by Λ. Then we can consider the subgroup $\widetilde{G}_{\Lambda,W}$ of $\widetilde{\mathrm{PL}}_+(\mathbf{S}^1)$ consisting of those elements with slopes in Λ and break-points in W (for instance, Thompson group is the case when Λ consists of powers of 2 and W of dyadic rationals). These groups are quite interesting especially when Λ is finitely generated (see [8, 9, 63]). It would be very useful to understand the nature of translation numbers of elements of $\widetilde{G}_{\Lambda,W}$ for specific Λ and W.

In [34], one can find (among other things!) a very interesting analysis of the rotation numbers of an explicit 1-parameter family of piecewise linear homeomorphisms of the circle.

5.2 TITS' ALTERNATIVE

Recall that J. Tits proved a remarkable alternative for finitely generated subgroups Γ of $GL(n, \mathbf{C})$ (see [65]): either Γ *contains a non abelian free subgroup or* Γ *contains a subgroup of finite index which is solvable.* Such an alternative does not hold for subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$. Indeed, we have seen that the group $\mathrm{PL}_+([0,1])$ can be considered as a subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and contains no free non abelian subgroup. The subgroup F of $\mathrm{PL}_+([0,1])$ consisting of elements whose slopes are powers of 2 and whose break-points are dyadic rationals, is a finitely presented group and is certainly not virtually solvable (since its first commutator subgroup is a simple group, see [28]). However, answering a question of the author, Margulis recently proved the following theorem [49]:

THEOREM 5.14 (Margulis). *Let* Γ *be a subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *At least one of the following properties holds:*

- Γ *contains a non abelian free subgroup.*
- *There is a probability measure on the circle which is* Γ*-invariant.*

COROLLARY 5.15. *Let* Γ *be a subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *such that all orbits are dense in the circle. Exactly one of the following properties holds:*

- Γ *contains a non abelian free subgroup.*
- Γ *is abelian and is conjugate to a group of rotations.*

The corollary follows easily from the theorem. Indeed, if all Γ-orbits are dense, any invariant probability must have full support and cannot have any non trivial atom. Any such probability is the image of the Lebesgue measure by some homeomorphism of the circle. Hence, up to some conjugacy, one can assume that Γ preserves the Lebesgue measure, *i.e.* consists of rotations. Note however that the proof which follows will begin with a proof of the corollary...

The proof of Margulis' theorem is very elegant and we cannot refrain from giving an account of it. Our presentation is a variation (or maybe a simplification ?) of Margulis' original ideas. More precise results may be found in the recent preprint [6]. We begin by recalling the "ping-pong" lemma, which is the standard way of constructing free subgroups (see [31]). Suppose a set X contains two disjoint non empty subsets A and A'. Let f, f' be two bijections of X which are such that for every $n \in \mathbf{Z} \setminus \{0\}$, we have $f^n(A) \subset A'$ and $f'^n(A') \subset A$. Then we claim that *f and f' generate a free subgroup of the group of bijections of X*. The proof is easy; consider a word $w(f,f') = f^{m_1} f'^{m'_1} \cdots f^{m_k} f'^{m'_k}$ with non zero exponents m_i, m'_i, except maybe the first one m_1 and the last one m'_k (if $k=1$, we assume that m_1 and m'_1 are not both zero...). We want to show that $w(f,f')$ represents a non trivial bijection of X. This is clear if $m_1 \neq 0$ and $m'_k = 0$ (resp. $m_1 = 0$ and $m'_k \neq 0$) since in this case we have $w(f,f')(A) \subset A'$ (resp. $w(f,f')(A') \subset A$). In the other cases, one can conjugate $w(f,f')$ by a suitable power of f or f' to get a new word which is in the previous form. This proves the ping-pong lemma.

In the case of the circle, the typical application of the ping-pong lemma is the following. Let I, J, I', J' be four closed intervals in the circle and let f,f' be two orientation preserving homeomorphisms of the circle. Assume the following condition holds:

(PING-PONG) The four intervals I, J, I', J' are disjoint, $f'(I) = \mathbf{S}^1 \setminus \mathit{interior}(J)$ and $f(I') = \mathbf{S}^1 \setminus \mathit{interior}(J')$.

Clearly, if one sets $X = \mathbf{S}^1$, $A = I \cup J$ and $A' = I' \cup J'$, we are in the situation of the ping-pong lemma and one can deduce from (PING-PONG) that f and f' generate a free subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$.

In order to find free subgroups inside a given subgroup Γ of $\mathrm{Homeo}_+(\mathbf{S}^1)$, we shall try to locate such ping-pong situations.

Assume now that we are given a group Γ such that the following two properties hold:

(MINIMALITY) All Γ-orbits are dense.

(STRONG EXPANSIVITY) There is a sequence of closed intervals I_n in the circle and a sequence γ_n of elements of Γ such that the length of I_n tends to zero as well as the length of the complementary intervals $J_n = \mathbf{S}^1 \setminus \mathit{int}(\gamma_n(I_n))$.

Of course, using subsequences we can assume in (STRONG EXPANSIVITY) that both endpoints of I_n converge to some point x and that both endpoints of J_n converge to some point y. We can also assume that $x \neq y$, since otherwise we could replace γ_n by $\gamma\gamma_n$ where γ is some element of Γ such that $y = \gamma(x) \neq x$.

Choose some γ in Γ such that $x' = \gamma(x)$ and $y' = \gamma(y)$ are both different from x and y (exercise: show that such an element γ exists!) and consider the sequence $\gamma'_n = \gamma^{-1}\gamma_n\gamma$. Of course, if we let $I'_n = \gamma(I_n)$ (resp. $J'_n = \gamma(J_n)$), the sequence of intervals I'_n (resp. J'_n) shrinks to x' (resp. to y'). Clearly, if n is big enough, the four intervals $I = I_n, J = J_n, I' = I'_n, J' = J'_n$ and the two homeomorphisms $f = \gamma_n, f' = \gamma'_n$ satisfy (PING-PONG) and therefore γ_n

and γ_n' generate a free subgroup of Γ. *In other words, if* (MINIMALITY) *and* (STRONG EXPANSIVITY) *hold, then* Γ *contains a free non abelian subgroup.*

The minimality condition is not so restrictive: we saw earlier that any action without a finite orbit is semi-conjugate to such a minimal action. However, the strong expansivity condition is very restrictive. Let us introduce the following weaker condition.

(EXPANSIVITY) There is a sequence of closed intervals I_n and a sequence of elements γ_n of Γ such that the length of I_n tends to zero and the length of $\gamma_n(I_n)$ is bounded away from zero.

Call a closed interval K in the circle *contractible* if there is a sequence of elements γ_n of Γ such that the length of $\gamma_n(K)$ tends to zero. It follows from (EXPANSIVITY) that there exists a non trivial contractible interval. If moreover the condition (MINIMALITY) is also satisfied, then every point of the circle belongs to the interior of some contractible interval. So let us assume now that (MINIMALITY) and (EXPANSIVITY) are satisfied.

For each point x in the circle, consider the set of points y such that the interval $[x,y]$ is contractible. Denote by $\theta(x)$ the least upper bound of those points y (to be correct, one should lift everything to the universal cover). In this way, we get a map θ from the circle to itself. Note that obviously θ commutes with all elements of Γ. Note also that θ is monotone. We claim that θ is a homeomorphism. Indeed if it were not strictly monotone, the union *Plat*(θ) of the interiors of the intervals in which θ is constant would be a Γ-invariant open set. By (MINIMALITY), this open set is empty unless θ is constant, but this is of course not possible since this constant would be fixed by Γ. In the same way, one shows that θ is continuous, using the union *Jump*(θ) of the interiors of the "jump intervals" like in 3.2.

We now consider the rotation number of θ. If this rotation number is irrational, then θ has to be conjugate to an irrational rotation since otherwise its unique invariant minimal set would be a non trivial Γ-invariant compact set. Since a homeomorphism which commutes with an irrational rotation is itself a rotation, that would imply that Γ is conjugate to a group of rotations. This is in contradiction with (EXPANSIVITY).

Hence the rotation number of θ is rational. The union of periodic points of θ is a non empty closed set which is Γ-invariant. It follows that θ is a periodic homeomorphism.

Consider the quotient $\mathbf{S}^{1'} = \mathbf{S}^1/\theta$ of the circle by the finite cyclic group generated by θ. This is a ("shorter") circle on which we have a natural action of Γ since, once again, Γ commutes with θ.

We observe that this new group of homeomorphisms of a circle satisfies (MINIMALITY) and (STRONG EXPANSIVITY). Minimality is obviously inherited from the same property of Γ on $\mathbf{S}^1$. As for (STRONG EXPANSIVITY), it suffices to observe that any compact interval contained in $[x, \theta(x)[$ is contractible, by definition. This means that any compact interval in $\mathbf{S}^{1'}$ is contractible and this implies (STRONG EXPANSIVITY).

We have now proved that if (MINIMALITY) *and* (EXPANSIVITY) *are both satisfied, then the group* Γ *must contain a free non abelian subgroup.*

Now, let us look more closely at (EXPANSIVITY) and observe that the negation of this property is nothing more than the *equicontinuity* property of the group Γ. If a group Γ acts equicontinuously, then its closure in $\mathrm{Homeo}_+(\mathbf{S}^1)$ is a compact group by Ascoli's theorem. We analyzed compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$ in 4.1 : they turned out to be abelian and conjugate to groups of rotations.

We have shown that if (MINIMALITY) *holds then* Γ *is either abelian or contains a free non abelian subgroup; in other words, we have proved Corollary 5.15.*

Proving Theorem 5.14 in full generality is now an easy matter. Let Γ be any subgroup of $\mathrm{Homeo}_+(\mathbf{S}^1)$ and let us use the structure theorem 5.6–5.8. If Γ is minimal, we have already proved the theorem. If Γ has a finite orbit, there is a Γ-invariant probability which is a finite sum of Dirac masses. Finally, if there is an exceptional minimal set, the Γ-action is semi-conjugate to a minimal action. Applying our proof to this minimal action, we deduce that Γ contains a non abelian free subgroup unless the restriction of the action of Γ to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations. In this case, one finds a Γ-invariant measure whose support is the exceptional minimal set. This is the end of the proof of Theorem 5.14.

6. BOUNDED EULER CLASS

6.1 GROUP COHOMOLOGY

Let us begin this section with some algebra. Let Γ be any group. Let us consider the (semi)-simplicial set $E\Gamma$ whose vertices are the elements of Γ and for which n-simplices are all $(n+1)$-tuples of elements of Γ. The i^{th} face of the simplex $(\gamma_0, \dots, \gamma_k)$ is $(\gamma_0, \dots, \hat{\gamma}_i \dots \gamma_k)$ where the term γ_i is omitted. Note that the set $E\Gamma$ does not depend on the group structure of Γ.