

## 5. Proof of the theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

of  $S$ . In particular, the cardinality of the quotient of the isometry group of  $S$  under the subgroup fixing a given systole equals  $6g + 3$ .

To complete the proof of our proposition we have to investigate the ideal surfaces  $S_\infty$  associated to simple triangle surfaces  $S(p; k)$ . The above considerations are equally valid for these surfaces and show that  $S_\infty$  has more than  $4g + 4$  systoles if and only if  $p$  divides  $k(k - 1) + 1$  and if the length  $\ell_0$  of a lift of a side pairing orbit for  $S_\infty$  is not bigger than  $6 \operatorname{arccosh} \frac{3}{2}$ . An explicit computation shows as before that this is the case if and only if  $S_\infty$  is associated to one of the surfaces  $S(7; 3), S(13; 4), S(21; 5)$ .  $\square$

## 5. PROOF OF THE THEOREM

Using the notation of Lemma 2.2, our goal is to show that the triangle surfaces  $S(7; 3), S(13; 4), S(21; 5)$  and their associated ideal surfaces are maximal. Following Schmutz [S1], for this it is enough to show that for each of these surfaces  $S$  the Teichmüller space is parametrized in a neighborhood of  $S$  by the lengths of those closed geodesics which are freely homotopic to a systole on  $S$ .

Let for the moment  $p \geq 5$  be an arbitrary odd number and let  $k \in \{2, \dots, p - 1\}$  be such that  $k$  and  $k - 1$  are prime to  $p$ . Write  $g = (p - 1)/2$ . As in the introduction let  $\mathcal{T}_{g,3}$  be the Teichmüller space of surfaces of genus  $g$  with 3 punctures. Let  $S = S(p; k)$  and let  $S_\infty$  be the ideal surface associated to  $S$ . The basic group  $\Gamma$  of orientation preserving isometries of  $S$  acts as a group of isometries on the surface  $S_\infty$ .

It will be useful to give a geometric description of  $S_\infty$ . For this let  $\Delta_\infty$  be an ideal triangle in  $\mathbf{H}^2$  and let  $T \subset \Delta_\infty$  be the finite equilateral triangle inscribed in  $\Delta_\infty$  which is invariant under all isometries of  $\Delta_\infty$ . The vertices of  $T$  determine a distinguished point on each side of  $\Delta_\infty$ .

There is a unique way to glue  $2p$  copies of  $\Delta_\infty$  to a disc  $A$  with one puncture in its interior and  $2p$  punctures on the boundary in such a way that the glueing maps identify the distinguished points on the sides of  $\Delta_\infty$ . The boundary of  $A$  then consists of  $2p$  geodesic lines. Each of the triangles which makes up  $A$  contains exactly one of these boundary geodesics. We number the boundary geodesics in counter clockwise order and glue the  $2i + 1$ -th geodesic to the  $2i + 2k$ -th geodesic by an orientation reversing isometry which identifies the distinguished points on these geodesics. The resulting surface is the ideal surface  $S_\infty$  associated to  $S$ . Notice that  $S_\infty$  admits a canonical triangulation into ideal triangles which corresponds to the canonical triangulation of  $S$ .

Denote by  $0, 1, 2$  the cusps of  $S_\infty$ . There are  $p$  edges of the canonical triangulation which connect the cusp  $0$  to the cusp  $1$ . There is a natural counter clockwise ordering of these edges which is induced by the ordering of the ideal triangles around the cusp  $0$ . We denote by  $\alpha_i^0$  the  $i$ -th edge with respect to this ordering and orient it in such a way that it goes from  $0$  to  $1$ . Similarly we define  $\alpha_i^1$  to be the  $i$ -th edge of our triangulation with respect to the counter-clockwise ordering around the cusp  $1$  which goes from the cusp  $1$  to the cusp  $2$ . Let also  $\alpha_i^2$  be the  $i$ -th edge ordered around the cusp  $2$  which goes from the cusp  $2$  to the cusp  $0$ .

Each marked surface of genus  $g = (p - 1)/2$  with three punctures can be triangulated by  $2p$  ideal triangles. If we cut the surface open along the edges of this triangulation, then we obtain  $2p$  ideal triangles. To get the surface back we glue the triangles along their boundary geodesics in the fixed combinatorial pattern as above. The different points in  $\mathcal{T}_{g,3}$  then differ by the way this glueing is arranged.

Namely, for each glueing we have one degree of freedom which corresponds to a left earthquake path along one of the geodesic arcs  $\alpha_i^j$ . Using the marking given by the distinguished points on the boundary of an ideal triangle and the induced boundary orientation, the glueings of an ordered pair  $(T_1, T_2)$  of (oriented) ideal triangles along a boundary geodesic can be parametrized by a real (left) sliding parameter. The glueing which identifies the distinguished points corresponds to the parameter  $0$ . A glueing where the distinguished point on the boundary geodesic of the triangle  $T_1$  is mapped to the right of the distinguished point on the boundary geodesic of the triangle  $T_2$  corresponds to a positive sliding parameter.

Following Thurston [T], in order to obtain a complete hyperbolic surface from the  $3p$  glueings of  $2p$  ideal triangles in the above combinatorial way, it is necessary and sufficient that at each of the three punctures of the resulting surface the sum of all the sliding parameters for all geodesics which go to this puncture vanishes. Thus if we denote by  $V \subset \mathbf{R}^p$  the linear subspace of all vectors which are orthogonal to the vector  $(1, \dots, 1)$ , then there is a natural bijection of  $\mathcal{T}_{g,3}$  onto  $V^3 = V \times V \times V$  which maps a surface  $M \in \mathcal{T}_{g,3}$  to its  $3p$ -tuple of sliding parameters.

Let now  $\gamma_i^0$  be the piecewise geodesic in  $S_\infty$  which consists of the arc  $\alpha_i^0$  with the orientation reversed and the arc  $\alpha_{i+k}^0$ . If we compactify the surface  $S_\infty$  by adding a point at each puncture, then the compactification of  $\gamma_i^0$  is a simple closed curve on  $S = S(p; k)$  which is freely homotopic to the closed geodesic  $\psi_i^0$  on  $S$  obtained by projecting a geodesic in a fundamental  $2p$ -gon  $\Omega$  which connects the midpoints of the edges  $2i + 1$  and  $2i + 2k$ . Similarly,

let  $k(1), k(2) \in \{2, \dots, p-1\}$  be such that  $k(1)(k-1) + 1 \equiv 0 \pmod{p}$  and  $k(k(2)-1) + 1 \equiv 0 \pmod{p}$  and denote for  $j = 1, 2$  by  $\gamma_i^j$  the piecewise geodesic which consists of the geodesic  $\alpha_i^j$  with the reversed orientation and the geodesic  $\alpha_{i+k(j)}^j$ . Write also  $k(0) = k$ .

An earthquake path through  $S_\infty$  induced by the curve  $\gamma_i^j$  deforms the surface  $S_\infty$  by a family of glueings with sliding parameter  $-t$  along  $\alpha_i^j$ , sliding parameter  $t$  along  $\alpha_{i+k(j)}^j$  ( $t \in \mathbf{R}$ ) and sliding parameter 0 otherwise and hence this earthquake path gives rise to a smooth (in fact real analytic) curve in  $\mathcal{T}_{g,3}$ . From this observation the following is immediate.

**LEMMA 5.1.** *For every surface  $M \in \mathcal{T}_{g,3}$  the tangents of the earthquake paths along the curves  $\gamma_i^j$  span the tangent space of  $\mathcal{T}_{g,3}$  at  $M$ .*

*Proof.* Let  $M \in \mathcal{T}_{g,3}$  and denote by  $\xi_i^j$  the tangent at  $M$  of the earthquake path along  $\alpha_i^j$ . We observed above that there is a linear isomorphism of the vector space  $V^3$  onto the tangent space of  $\mathcal{T}_{g,3}$  at  $M$  which maps the point  $(0_1, \dots, 0_p, a_1, \dots, a_p, b_1, \dots, b_p) \in V^3$  to the tangent vector  $\sum_{i,j} j_i \xi_i^j$ . Since the tangent at  $M$  of the earthquake path induced by  $\gamma_i^j$  is just  $\xi_{i+k(j)}^j - \xi_i^j$  the lemma follows.  $\square$

There is a natural real analytic submersion  $P$  of  $\mathcal{T}_{g,3}$  onto  $\mathcal{T}_g$  which is equivariant under the action of the basic group  $\Gamma$ . This submersion simply maps a surface of genus  $g$  with 3 punctures to the surface obtained by compactifying each puncture with a single point. For every  $S \in \mathcal{T}_g$  the fibre of  $P$  over  $S$  consists of all surfaces in  $\mathcal{T}_{g,3}$  which we obtain from  $S$  by removing an ordered triple of pairwise distinct points. In particular, the fibre is a real analytic submanifold of  $\mathcal{T}_{g,3}$  of dimension 6. We denote by  $W$  the 6-dimensional subbundle of the tangent bundle of  $\mathcal{T}_{g,3}$  which is the kernel of the differential of  $P$ . This bundle has a natural direct decomposition  $W = W_0 \oplus W_1 \oplus W_2$  into two-dimensional subbundles  $W_j$ . Here the bundle  $W_j$  is the tangent bundle of the fibres of the fibration  $\mathcal{T}_{g,3} \rightarrow \mathcal{T}_{g,2}$  which we obtain by adding for every surface  $M \in \mathcal{T}_{g,3}$  a single point at the puncture  $j$  of  $M$ .

For  $M \in \mathcal{T}_{g,3}$  the compactifications of the curves  $P\gamma_i^j$  are homotopically nontrivial simple closed curves on  $PM$ . There is a unique free homotopy class on  $M$  which can be represented by a closed curve which does not intersect  $\gamma_i^j$  and whose projection to  $PM$  is freely homotopic to the compactification of  $P\gamma_i^j$ . We denote by  $\tilde{\psi}_i^j$  the unique geodesic on  $M$  representing this class. We have.



LEMMA 5.2. *Let  $\xi_i^j, \zeta_i^j$  be the tangent of the earthquake path along  $\tilde{\psi}_i^j, \gamma_i^j$ . Then there are functions  $a_i^j: \mathcal{T}_{g,3} \rightarrow \mathbf{R}$  such that  $\zeta_i^j - a_i^j \xi_i^j \in W_j \oplus W_{j+1}$ .*

*Proof.* Let  $M \in \mathcal{T}_{g,3}$  and for  $i \in \{1, \dots, p\}, j = 0, 1, 2$  consider the piecewise geodesic  $\gamma_i^j$  and the geodesic  $\tilde{\psi}_i^j$  on  $M$ . Since the number of intersections between  $\gamma_i^j$  and  $\tilde{\psi}_i^j$  is the minimum of the number of intersections between  $\gamma_i^j$  and any curve which is freely homotopic to  $\tilde{\psi}_i^j$ , the geodesics  $\tilde{\psi}_i^j$  and  $\gamma_i^j$  on  $M$  do not intersect. If we cut the surface  $M$  open along the curves  $\gamma_i^j$  and  $\tilde{\psi}_i^j$  then the interior of one of the connected surfaces with boundary which we obtain in this way, say the surface  $C$ , is homeomorphic to an open annulus. One boundary component of  $C$  is the curve  $\tilde{\psi}_i^j$ , the second boundary component has two punctures and consists of the curve  $\gamma_i^j$ .

By construction, the curve  $\tilde{\psi}_i^j$  is non-separating and therefore there is a simple closed geodesic  $\eta$  on  $M$  which neither intersects  $\gamma_i^j$  nor  $\tilde{\psi}_i^j$  and such that after cutting  $M$  along  $\eta$  we obtain two bordered surfaces  $S_1, S_2$ . The surface  $S_1$  is a surface of genus 1 with one geodesic boundary circle and two punctures in its interior and contains the annulus  $C$  bounded by the curves  $\gamma_i^j$  and  $\tilde{\psi}_i^j$ . The earthquake paths along the piecewise geodesic  $\gamma_i^j$  and the geodesic  $\tilde{\psi}_i^j$  leave the hyperbolic length of a closed geodesic  $\sigma$  on  $M$  fixed if and only if  $\sigma$  does not have a transverse intersection with  $\gamma_i^j, \tilde{\psi}_i^j$ . Thus these earthquake paths define deformations of the hyperbolic structure on  $S_1$  leaving the length of the boundary fixed.

The Teichmüller space of marked hyperbolic structures on the bordered torus  $S_1$  with two punctures and a boundary geodesic of fixed length is 6-dimensional. Its tangent bundle contains a 5-dimensional subbundle  $V$  which consists of all infinitesimal deformations preserving the modulus of a maximal (twice punctured) ring domain with core curve homotopic to  $\tilde{\psi}_i^j$ .

We claim that this 5-dimensional subbundle  $V$  contains the tangents of the earthquake paths along the geodesic  $\tilde{\psi}_i^j$  and along the piecewise geodesic  $\gamma_i^j$ .

To see this let  $\zeta$  be the unique simple geodesic arc in  $S_1$  which meets the boundary geodesic  $\eta$  perpendicularly and which neither intersects  $\tilde{\psi}_i^j$  nor  $\gamma_i^j$ . Let  $\bar{S}_1$  be the compactification of  $S_1$  which we obtain by simply adding one point at each puncture. If we cut  $\bar{S}_1$  open along  $\zeta$ , then we obtain a standard ring domain  $A$  normalized by the fixed choice of a height, say the height 1, with core curve homotopic to  $\tilde{\psi}_i^j$  and whose modulus is maximal among all ring domains with this property [St]. The boundary of  $A$  consists of two circles which contain each a copy of the arc  $\zeta$  as well as a nontrivial component of the boundary geodesic  $\eta$ . We mark the arc on each boundary component which corresponds to the arc  $\zeta$ . The surface  $\bar{S}_1$  is obtained by glueing the

two marked arcs on the two boundary components with the restriction of a complex linear map of the complex plane.

The compactification of  $\gamma_i^j$  is a closed curve in the ring domain  $A$  which is freely homotopic to the core curve. If we cut  $A$  open along this curve then by uniformization we obtain two standard ring domains  $A_1, A_2$  with one common boundary circle. The earthquake path induced by  $\gamma_i^j$  consists in cutting  $A$  along the common boundary circle of  $A_1, A_2$  and glueing the ring domains  $A_1, A_2$  back with a new boundary identification. This procedure does not change the lengths of the arcs  $\eta$  or  $\zeta$  nor the modulus of the annulus  $A$ . In other words, the tangent of this earthquake path is contained in  $V$ . The same argument applies to the earthquake path induced by the geodesic  $\tilde{\psi}_i^j$ . We conclude that this earthquake path induces a nontrivial infinitesimal deformation of the conformal structure on the compactification of our bordered punctured torus which leaves the modulus of a maximal ring domain with core curve homotopic to  $\tilde{\psi}_i^j$  fixed. In particular, the tangent of this earthquake path is contained in  $V$  but not in the kernel of the differential of the natural map which assigns to a twice punctured bordered torus its compactification.

As a conclusion, the tangent at  $M$  of the earthquake path induced by  $\gamma_i^j$  can be written in the form  $a_i^j \xi_i^j + \eta_i^j$  where  $\xi_i^j$  is the tangent of the earthquake path along  $\tilde{\psi}_i^j$ ,  $a_i^j \in \mathbf{R}$  and  $\eta_i^j$  is contained in the bundle  $W_j \oplus W_{j+1}$ . This shows the lemma.  $\square$

Let now  $k \geq 3$  and consider again the ideal surface  $S_\infty$  associated to the simple triangle surface  $S = S(k(k-1)+1; k)$ . Using the above notation, for  $m = jp + i$  ( $j \in \{0, 1, 2\}, i < p$ ) write  $\tilde{\psi}_m = \tilde{\psi}_i^j$ . For  $M \in \mathcal{T}_{g,3}$  and  $m \in \{1, \dots, 3p\}$  denote by  $\ell_M(\tilde{\psi}_m)$  the length of the closed geodesic  $\tilde{\psi}_m$  on  $M$ . The functions  $M \in \mathcal{T}_{g,3} \rightarrow \ell_M(\tilde{\psi}_m)$  are real analytic [K]. This means that we obtain a real analytic map  $\Psi_\infty$  of  $\mathcal{T}_{g,3}$  into  $\mathbf{R}^{3p}$  by mapping a surface  $M$  to  $\Psi_\infty(M) = (\ell_M(\tilde{\psi}_1), \dots, \ell_M(\tilde{\psi}_{3p}))$ . From Lemma 5.1 and Lemma 5.2 we conclude.

**COROLLARY 5.3.** *The map  $\Psi_\infty$  is of maximal rank differentiable at  $S_\infty$ .*

*Proof.* Following Wolpert [W], the tangent of the earthquake path along  $\tilde{\psi}_i^j$  is dual with respect to the Weil Petersen Kähler form to the differential of the length function of  $\tilde{\psi}_i^j$  on  $\mathcal{T}_{g,3}$ . Thus to show the corollary it is enough to show that the tangent space of  $\mathcal{T}_{g,3}$  at  $S_\infty$  is spanned by the tangents  $\xi_i^j$  of the earthquake paths along the curves  $\tilde{\psi}_i^j$ .

Let  $G$  be the group of isometries of  $S_\infty$  which is generated by the basic group  $\Gamma$  and the group  $\Sigma$  of order 3 contained in the normalizer of  $\Gamma$ . The group  $G$  acts on the Teichmüller space  $\mathcal{T}_{g,3}$  as a group of automorphism which fixes the surface  $S_\infty$ .

Let  $\Lambda$  be the linear isometry of  $\mathbf{R}^p$  defined in canonical coordinates by  $\Lambda(x_1, \dots, x_p) = (x_2, \dots, x_p, x_1)$ ; then  $\Lambda \times \Lambda \times \Lambda = \Lambda_3$  is a linear isometry of  $\mathbf{R}^{3p}$ . If  $J_1$  is the canonical generator of the normal cyclic subgroup  $\Gamma$  of  $G$  then we have  $\Psi_\infty(J_1 M) = \Lambda_3 \Psi_\infty(M)$ .

Let  $\tau$  be the linear isometry of  $\mathbf{R}^{3p} = \mathbf{R}^p \times \mathbf{R}^p \times \mathbf{R}^p$  which cyclicly permutes the factors  $\mathbf{R}^p$  in the direct decomposition of  $\mathbf{R}^{3p}$ . There is a permutation  $\sigma$  of  $\{1, \dots, p\}$  of order  $p-1$  with diagonal extension  $\sigma_3$  to  $\mathbf{R}^{3p}$  such that the canonical generator  $J_2$  of the cyclic subgroup  $\Sigma$  acts by  $\Psi_\infty J_2(M) = \sigma_3 \circ \tau(\Psi_\infty M)$ .

The eigenvalues of the linear isometry  $\Lambda$  are the  $p$ -th roots of unity. The eigenspace for the eigenvalue 1 is spanned by  $(1, \dots, 1)$  and the other generalized eigenspaces are of dimension 2. The map  $\sigma_3 \circ \tau$  permutes the generalized eigenspaces of the diagonal extension  $\Lambda_3$  which correspond to eigenvalues different from 1 and acts as a cyclic group of permutations on the eigenspace  $Z$  of  $\Lambda^3$  with respect to the eigenvalue 1. The orthogonal complement  $Z^\perp$  of  $Z$  in  $\mathbf{R}^{3p}$  decomposes into  $g$  irreducible invariant subspaces of dimension 6 each.

The surface  $S_\infty$  is a fixed point for the action of  $G$ . By Lemma 5.1, the tangent space of  $\mathcal{T}_{g,3}$  at  $S_\infty$  as a  $G$ -space is isomorphic to  $Z^\perp$ , where the differential of  $J_1$  acts as the map  $\Lambda_3$  and the differential of  $J_2$  as  $\sigma_3 \circ \tau$ . The 6-dimensional tangent space  $W$  at  $S_\infty$  of the fibre of the fibration  $P: \mathcal{T}_{g,3} \rightarrow \mathcal{T}_g$  is invariant under the action of  $G$  and for reasons of dimension necessarily irreducible.

Let as before  $\xi_i^j, \zeta_i^j$  be the tangent at  $S_\infty$  of the earthquake path along  $\tilde{\psi}_i^j, \gamma_i^j$ .

Denote by  $L$  the linear map which maps  $\zeta_i^j$  to  $\xi_i^j$ . Then  $L$  is  $G$ -equivariant and by Lemma 5.2 its kernel is contained in the  $G$ -invariant space  $W$ . Since  $W$  is irreducible under  $G$  the kernel of  $L$  is either trivial or coincides with  $W$ .

We have to show that the latter does not hold. For this we have to find a tangent vector  $X \in W$  such that  $LX \neq 0$ .

Consider the unit disc  $D$  in the complex plane with boundary circle  $S^1$  and hyperbolic metric. Let  $D_\infty$  be the disc with the point 0 deleted. It carries a unique complete hyperbolic metric for which the puncture is a standard cusp. This metric admits an isometric circle action which induces the standard parametrization of the boundary circle  $S^1 = [0, 2\pi)$ .

Let  $\Omega_0, \Omega$  be the regular ideal hyperbolic  $2p$ -gon in  $D_\infty, D$  whose set  $\mathcal{P}$  of vertices consists of the points  $j\pi/p$  ( $j = 1, \dots, 2p$ ). These  $2p$ -gons admit a cyclic group of order  $2p$  of isometries, and  $\Omega_0$  hence is isometric to the once punctured polygon which we obtain by cutting  $S_\infty$  along the geodesics of the canonical triangulation joining the cusps 1 and 2.

For an interior point  $x$  of  $\tilde{\Omega}$  consider the polygon  $\Omega_x = \tilde{\Omega} \setminus \{x\}$  with one puncture at  $x$ . The punctured polygon  $\Omega_x$  carries a hyperbolic metric of finite volume such that the boundary consists of  $2p$  geodesic lines, and it is naturally triangulated into  $2p$  ideal triangles.

Let  $\gamma$  be a hyperbolic geodesic in  $D$  through  $\gamma(0) = 0$ . For every  $t \in \mathbf{R}$  there is a unique hyperbolic isometry  $\Psi_t$  of  $D$  which fixes the endpoints of  $\gamma$  and maps  $\gamma(t)$  to 0. The image under  $\Psi_t$  of the punctured polygon  $\Omega_{\gamma(t)}$  is an ideal hyperbolic polygon with puncture at 0 and whose vertices on  $S^1$  are the points in  $\Psi_t \mathcal{P}$ . The punctured polygon  $\Psi_t \Omega_{\gamma(t)}$  can be obtained from  $\Omega_0$  by an earthquake deformation along the geodesics which joins 0 to the vertices of  $\Omega_0$  as follows.

Consider an ordered triple  $(a, b, c)$  of 3 pairwise distinct points on the boundary circle  $S^1$  of  $D_\infty$  arranged in counter clockwise order. These points determine an ideal quadrangle  $Q$  which decomposes  $Q$  into 2 ideal hyperbolic triangles embedded in  $D_\infty$  which have one vertex at 0. Let  $T_1$  be the triangle with vertices  $a, b$ , and let  $T_2$  be the triangle with vertices  $b, c$ . If the euclidean distance between  $a$  and  $b$  is smaller than the distance between  $b$  and  $c$  then the glueing map which gives the quadrangle  $Q$  back from the triangles  $T_1$  and  $T_2$  maps the distinguished point of the boundary geodesic of  $T_1$  to the right of the distinguished point on the boundary geodesic of  $T_2$  with respect to the boundary orientation of  $T_2$ . In other words, with our above notation the glueing corresponds to a positive sliding parameter.

The derivative of the restriction of  $\Psi_t$  to  $S^1$  has a maximum at its repelling fix point  $z_1$  and a minimum at its attracting fix point  $z_2$ . It is strictly monotonous on each of the two components of  $S^1 - \{z_1, z_2\}$ . Let  $(z_1, z_2)$  be the component which corresponds to an open interval in  $[0, 2\pi)$  with left endpoint  $z_1$ . The above analysis shows that the deformation of the polygon  $\Omega_0$  which defines  $\Psi_t \Omega_{\gamma(t)}$  has a negative sliding parameter for every geodesic which joins 0 to a point in  $\mathcal{P} \cap (z_1, z_2)$ . The sliding parameter is positive for all geodesics which join 0 to a point in  $\mathcal{P} \cap (z_2, z_1)$ .

Choose now  $\gamma$  in such a way that its forward endpoint equals  $k\pi/2p$  and that its backward endpoint equals  $k\pi/2p + \pi$ . Let  $\rho$  be the reflection of  $\tilde{\Omega}$  along  $\gamma$ . This reflection induces an orientation reversing isometry of  $D_\infty$  which commutes with the above deformation of  $\Omega_0$  along  $\gamma$ . Denote by  $\beta_i$

the geodesic which connects the center 0 to  $(k+i)\pi/2p$  ( $1 \leq i \leq 2p$ ) and let  $\nu_i$  be the tangent of the earthquake path along  $\beta_i$ . By symmetry, the tangent at  $t = 0$  of our deformation of  $\Omega_0$  along  $\gamma$  can be written in the form  $\sum a_i \nu_i$  where  $a_i < 0$  and  $a_{i-p} = -a_i$  for  $i = 1, \dots, p-1$ .

Consider now the geodesic  $\tilde{\psi}_1^0$ . It intersects  $\gamma$  perpendicularly and has  $2k-2 \geq 2$  additional intersections with the geodesics  $\beta_i$ . For  $i \in \{1, \dots, k-1\}$  denote by  $\delta_i$  the oriented angle of the intersection of  $\tilde{\psi}_1^0$  with the geodesic  $\beta_i$ , where we write  $\delta_i = \pi/2$  if the geodesics  $\beta_i$  and  $\gamma$  do not intersect. By invariance under  $\rho$  we have  $\delta_{2p-i} - \pi/2 = -(\delta_i - \pi/2)$ .

Following Kerckhoff (see [K]), the derivative at  $t = 0$  of the length of  $\tilde{\psi}_1^0$  under our deformation of  $\Omega_0$  equals up to a positive constant the sum  $\sum a_i \cos \delta_i$ . But  $0 > \cos \delta_i = -\cos \delta_{2p-i}$  for  $1 \leq i \leq k-1$  and  $\cos \delta_i = 0$  otherwise and therefore the derivative of the length of  $\tilde{\psi}_1^0$  under our above deformation of  $\Omega_0$  does not vanish. In other words, the map  $L$  does not vanish on  $W$ . This completes the proof of the corollary.  $\square$

Let now again  $p \geq 5$  be arbitrary and write  $g = (p-1)/2$ . Using the above notation, for  $M \in \mathcal{T}_{g,3}$  let  $\psi_i^j$  be the closed geodesic on the surface  $PM$  which is freely homotopic to the compactification of the curve  $\gamma_i^j$ . For  $S \in \mathcal{T}_g$  let  $\ell_S(\psi_i^j)$  be the length of  $\psi_i^j$ . We then obtain a real analytic map  $\Psi$  of  $\mathcal{T}_g$  into  $\mathbf{R}^{3p}$  by mapping  $S$  to  $\Psi(S) = (\ell_S(\psi_1^0), \dots, \ell_S(\psi_p^2))$ .

Theorem *B* from the introduction is an immediate consequence of the following.

**LEMMA 5.4.** *The map  $\Psi$  is of maximal rank differentiable and injective.*

*Proof.* Let again  $\xi_i^j$  be the tangent of the earthquake path along the closed geodesic  $\psi_i^j$ . By the results of Wolpert [W] it suffices to show that the tangent space of  $\mathcal{T}_g$  at any point  $S$  is spanned by the vectors  $\xi_i^j$ .

An arbitrary choice of three points in the complement of the curves  $\psi_i^j$  on  $S$  defines a surface  $M \in \mathcal{T}_{g,3}$ . The earthquake path in  $\mathcal{T}_g$  induced by  $\psi_i^j$  naturally lifts to a path in  $\mathcal{T}_{g,3}$ . The consideration in the proof of Lemma 5.2 shows that this lift is (up to parametrization and up to possibly moving the punctures) just the earthquake path in  $S_\infty$  along  $\tilde{\psi}_i^j \in M$ . This implies by Lemma 5.2 and Lemma 5.1 that the tangent space of  $\mathcal{T}_g$  at  $M$  is spanned by the vectors  $\xi_i^j$  and shows that  $\Psi$  is of maximal rank differentiable. Since the earthquake paths along the curves  $\gamma_i^j$  parametrize  $\mathcal{T}_{g,3}$  the map  $\Psi$  is moreover injective.  $\square$



The next corollary is an immediate consequence of Lemma 5.3, Lemma 5.4, Proposition 4.8 and the results of Schmutz in [S1].

**COROLLARY 5.5.** *The surfaces  $S(7;3)$ ,  $S(13;4)$ ,  $S(21;5)$  and their associated ideal surfaces are maximal.*

We conclude the paper with some remarks about the relation between our triangulation and the structure of the Thurston boundary of Teichmüller space.

Consider for the moment an arbitrary closed surface  $S$ . A *geodesic current* for  $S$  is a locally finite Borel-measure on the space of unoriented geodesics in the hyperbolic plane  $\mathbf{H}^2$  which is invariant under the action of the fundamental group  $\pi_1(S)$  of  $S$ . The space  $\mathcal{C}$  of geodesic currents for  $S$  only depends on the topological type of  $S$ . There is a bilinear form  $i$  on  $\mathcal{C}$ , the so called *intersection form*, which is continuous with respect to the weak\*-topology on  $\mathcal{C}$ . The subset  $\mathcal{L}$  of  $\mathcal{C}$  of all geodesic currents  $\mu$  with vanishing self-intersection  $i(\mu, \mu) = 0$  is the space of *measured geodesic laminations* and is homeomorphic to  $\mathbf{R}^{6g-6}$  [B].

Let  $\mathcal{PC}$  and  $\mathcal{PL}$  be the projectivization of the space of nonzero geodesic currents and laminations. There is a natural continuous embedding  $J$  of the Teichmüller space  $\mathcal{T}_g$  into  $\mathcal{PC}$  by mapping  $M \in \mathcal{T}_g$  to the projectivization  $[\lambda_M]$  of its Lebesgue-Liouville current  $\lambda_M$ . The closure of  $J(\mathcal{T}_g)$  in  $\mathcal{PC}$  is just  $J(\mathcal{T}_g) \cup \mathcal{PL}$  [B].

Every simple closed geodesic  $\psi$  on  $S$  can naturally be viewed as a measured geodesic lamination and hence induces a linear functional on  $\mathcal{C}$  via  $\mu \rightarrow i(\psi, \mu)$ . If  $\lambda_M$  is the Lebesgue-Liouville current of a point  $M \in \mathcal{T}_g$  in Teichmüller space then  $i(\lambda_M, \psi) = \ell_M(\psi)$  is just the  $M$ -length of  $\psi$  [B]. In particular, the map  $M \in \mathcal{T}_g \rightarrow i(\lambda_M, \psi)$  is real analytic.

Recall that a collection  $\psi_1, \dots, \psi_k$  of simple closed curves on  $S$  *fills up* if every geodesic on  $S$  intersects one of the curves  $\psi_i$  transversely. This is equivalent to saying that the complement of  $\{\psi_1, \dots, \psi_k\}$  in  $S$  consists of a finite collection of connected simply connected regions. If  $\psi_1, \dots, \psi_k$  fills up then for every measured geodesic lamination  $\mu \in \mathcal{L}$  the vector  $(i(\psi_1, \mu), \dots, i(\psi_k, \mu)) \in \mathbf{R}^k$  does not vanish. Thus if we denote by  $\mathbf{PR}^k$  the real projective space of all lines in  $\mathbf{R}^k$  and for  $0 \neq x \in \mathbf{R}^k$  by  $[x] \in \mathbf{PR}^k$  the line in  $\mathbf{R}^k$  through  $x$  then the map  $A: M \in \mathcal{T}_g \rightarrow [\ell_M(\psi_1), \dots, \ell_M(\psi_k)] \in \mathbf{PR}^k$  extends continuously to the Thurston compactification  $\mathcal{PL}$  of  $\mathcal{T}_g$  by mapping the projective class  $[\mu]$  of  $\mu \in \mathcal{L}$  to  $A([\mu]) = [i(\psi_1, \mu), \dots, i(\psi_k, \mu)]$ . A family  $(\psi_1, \dots, \psi_k)$  of simple closed curves on  $S$  is called *parametrizing* for

$\mathcal{PL}$  if the map  $[\mu] \in \mathcal{PL} \rightarrow A([\mu]) = [i(\psi_1, \mu), \dots, i(\psi_k, \mu)] \in \mathbf{PR}^k$  is an embedding.

It is also possible to define geodesic currents and measured geodesic laminations for hyperbolic surfaces with cusps. By definition, a measured geodesic lamination of such a surface  $M$  with cusps is a compact subset of  $M$  which is foliated by geodesics and equipped with a transverse invariant measure.

Let now  $p \geq 5$  and let  $k \in \{2, \dots, p-1\}$  be such that  $k$  and  $k-1$  are prime to  $p$ . Denote by  $S_\infty$  the ideal surface associated to the triangle surface  $S(k; p)$  and let  $\gamma_i^j$  the piecewise geodesics as in Lemma 5.1. If  $\psi$  is any closed geodesic in  $S_\infty$  then  $\psi$  does not disappear in the cusps of  $S_\infty$  and hence  $\psi$  intersects each of the geodesics  $\gamma_i^j$  transversely in a finite number of points. We denote by  $i(\psi, \gamma_i^j)$  the number of intersections of  $\psi$  with  $\gamma_i^j$ . Since measured laminations on  $S_\infty$  have compact support, intersection of closed geodesics with one of the curves  $\gamma_i^j$  extends to a continuous convex-linear functional  $i(\gamma_i^j, \cdot)$  on the space  $\mathcal{L}_\infty$  of measured geodesic laminations on  $S_\infty$ .

We have:

LEMMA 5.6. *The map  $\mu \in \mathcal{L}_\infty \rightarrow A(\mu) =$*

$$(i(\gamma_1^0, \mu), \dots, i(\gamma_p^0, \mu), i(\gamma_1^1, \mu), \dots, i(\gamma_p^1, \mu), i(\gamma_q^2, \mu), \dots, i(\gamma_p^2, \mu))$$

*is an embedding.*

*Proof.* It suffices to show that every closed geodesic  $\psi$  is determined by  $A(\psi)$ . For this consider again the edges  $\alpha_i^j$  of the canonical triangulation of  $S_\infty$ . It follows immediately from our construction that  $A(\psi)$  determines uniquely the tuple

$$C(\psi) = (i(\alpha_1^0, \psi), \dots, i(\alpha_p^0, \psi), i(\alpha_1^1, \psi), \dots, i(\alpha_p^1, \psi), i(\alpha_1^2, \psi), \dots, i(\alpha_p^2, \psi))$$

(compare the proof of Lemma 5.1). Thus it is enough to show that we can reconstruct  $\psi$  from  $C(\psi)$ .

The arcs  $\alpha_j^i$  define a triangulation of  $S_\infty$  into  $2p$  triangles with vertices at the cusps and such that each arc is the side of exactly two triangles. Let  $\psi$  be any closed geodesic on  $S_\infty$  and let  $T$  be a triangle from the triangulation with sides  $\beta_1, \beta_2, \beta_3$ . Write  $j_i = i(\beta_i, \psi)$  and assume that  $j_1 \geq j_2 \geq j_3$ . Since  $T$  is contractible in the compactification of  $S_\infty$ , the total intersection number  $j_1 + j_2 + j_3$  of  $\psi$  with the boundary of  $T$  is even and hence  $j_2 + j_3 - j_1$  is even as well. Moreover we have  $j_1 \leq j_2 + j_3$ . Draw  $\frac{1}{2}(j_2 + j_3 - j_1)$  simple arcs



connecting the sides  $\beta_2$  and  $\beta_3, j_2 - \frac{1}{2}(j_2 + j_3 - j_1)$  simple arcs connecting the sides  $\beta_1$  and  $\beta_2, j_3 - \frac{1}{2}(j_2 + j_3 - j_1)$  simple arcs connecting the sides  $\beta_1$  and  $\beta_3$  in such a way that all these arcs are disjoint. The configuration of these arcs in  $T$  is determined up to isotopy by  $j_1 \geq j_2 \geq j_3$ . But this means that  $\psi$  is uniquely determined by  $C(\psi)$  (compare the discussion in [FLP]) and hence the lemma follows.  $\square$

Recall that a closed curve  $\psi$  on  $S_\infty$  is *cuspidal-parallel* if  $\psi$  is homotopic to a multiple of a circle surrounding one of the cusps of  $S_\infty$ . This is equivalent to saying that the infimum of the lengths of all curves in  $S_\infty$  which are freely homotopic to  $\psi$  is zero (notice that by abuse of notation we call a contractible curve cuspidal-parallel as well). A closed curve  $\psi$  on  $S_\infty$  is freely homotopic to a closed geodesic if and only if  $\psi$  is not cuspidal-parallel.

We define now an equivalence relation on the set of all closed curves on  $S_\infty$  as follows: Let  $\psi, \eta: [0, 1] \rightarrow S_\infty$  be parametrized closed curves. Call  $\psi, \eta$  *equivalent* if there is a subdivision  $0 < t_1 < \dots < t_k < 1$  of  $[0, 1]$  and for each  $i$  there is a cuspidal-parallel loop  $\gamma_i$  through  $\psi(t_i)$  such that  $\eta$  is freely homotopic to  $\psi|_{[0, t_1]} \cup \gamma_1 \dots \cup \gamma_k \cup \psi|_{[t_k, 1]}$ .

This is easily seen to be an equivalence relation. The equivalence classes of this relation are in 1-1-correspondence to the free homotopy classes of closed curves on the surface  $S$ . We denote the class of  $\psi$  by  $[\psi]$ . For a closed curve  $\psi$  on  $S_\infty$  and for  $i \in \{1, \dots, p\}$ ,  $j = 0, 1, 2$  define  $\mathcal{J}(\psi, \gamma_i^j)$  to be the infimum of the number of intersections with  $\gamma_i^j$  of all curves  $\eta$  equivalent to  $\psi$ .

Let  $\psi_i^j$  be the closed geodesic on the surface  $S$  which is freely homotopic to the compactification of  $\gamma_i^j$  viewed as a curve on  $S$ . For every closed geodesic  $\eta$  on  $S$  which is different from a multiple of  $\psi_i^j$  the number of intersection points between  $\eta$  and  $\psi_i^j$  is the infimum  $i(\eta, \psi_i^j)$  of the number of intersection points between all curves freely homotopic to  $\eta, \psi_i^j$ .

We have:

LEMMA 5.7.  $\mathcal{J}(\zeta, \gamma_i^j) = i([\zeta], \psi_i^j)$  for every closed curve  $\zeta$  on  $S_\infty$ .

*Proof.* For every closed curve  $\zeta$  on  $S_\infty$  there is an equivalent curve  $\eta$  such that  $\mathcal{J}(\zeta, \gamma_i^j)$  equals the number of intersection points of  $\eta$  with  $\gamma_i^j$ . Now if we compactify  $S_\infty$  by adding a point at each cusp, then we obtain a surface  $M$  of genus  $g$  and  $\eta$  and  $\zeta$  are freely homotopic on  $M$ ,  $\gamma_i^j$  is freely homotopic to the curve  $\psi_i^j$ . But this means that  $\mathcal{J}(\zeta, \gamma_i^j) \geq i([\zeta], \psi_i^j)$ .

On the other hand, if  $\zeta$  is any closed curve on  $S$  with a minimal number of intersections with  $\psi_i^j$  in its free homotopy class, then we can remove from  $S$  three points which do not lie on  $\zeta$  and such that two of these points lie on  $\psi_i^j$ . If we call the resulting surface  $S_\infty$  then  $\zeta$  defines a closed curve  $\zeta_\infty$  on  $S_\infty$ , and  $i(\zeta, \psi_i^j)$  equals the number of intersection points between  $\zeta_\infty$  and  $\gamma_i^j$  (where  $\gamma_i^j$  is given as before). This then shows that  $\mathcal{J}(\zeta_\infty, \gamma_i^j) \leq i(\zeta, \psi_i^j) = i([\zeta_\infty], \psi_i^j)$   $\square$

As an immediate consequence of Lemma 5.6 and Lemma 5.7 we obtain

**COROLLARY 5.8.** *The curves  $\psi_i^j$  on  $S$  are parametrizing for  $\mathcal{PL}$ . In particular, for every  $g \geq 2$  there is a family of  $6g + 3$  free homotopy classes on a closed surface of genus  $g$  which is parametrizing for  $\mathcal{PL}$ .*

**REMARK.** From [FLP] one immediately obtains a family of  $9g - 9$  closed curves on a closed surface of genus  $g$  which is parametrizing for  $\mathcal{PL}$ . To my knowledge, the minimal number of simple closed curves with this property is not known.

**ACKNOWLEDGMENT.** This work was motivated by computer experiments using a computer program written by Roman Koch and was completed while I visited the IHES. I thank the Institute for its hospitality.

## REFERENCES

- [B] BONAHOE, F. The geometry of Teichmüller space via geodesic currents. *Invent. Math.* 92 (1988), 139–162.
- [Bu] BUSER, P. *Geometry and Spectra of Compact Riemann Surfaces*. Birkhäuser, Boston 1992.
- [BS] BUSER, P. and P. SARNAK. On the period matrix of a Riemann surface of large genus. *Invent. Math.* 117 (1994), 27–56.
- [FLP] FATHI, A., LAUDENBACH, F. and V. POÉNARU. Travaux de Thurston sur les surfaces. *Astérisque* 66–67 (1979).
- [F] FORSTER, O. *Lectures on Riemann Surfaces*. Springer Graduate Texts in Math. 81, New York, 1981.
- [G] GARDINER, F. *Teichmüller Theory and Quadratic Differentials*. Wiley, New York, 1987.
- [I] IVERSEN, B. *Hyperbolic Geometry*. Cambridge University Press, 1992.
- [K] KERCKHOFF, S. Lines of minima in Teichmüller space. *Duke Math. J.* 65 (1992), 182–213.