6. The cones of the products $S^n \times S^{2m-1}$

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iii) Let X and Y be the Moore spaces $M(\mathbf{Z}/3, 2q+11) = S^{2q+11} \cup_3 e^{2q+12}$ and $M(\mathbf{Z}/3, 2q-1) = S^{2q-1} \cup_3 e^{2q}$ respectively. In [Adams], Adams shows that for q large enough, there exists a map $A\colon X = \Sigma^{12} Y \longrightarrow Y$ such that the induced map $A^*\colon \widetilde{K}(Y) \longrightarrow \widetilde{K}(X)$ is an isomorphism (take $p = m = 3$, $f = 1$ and $r = 6$ in Theorem 1.7 and in Lemmas 12.4 and 12.5 of [Adams]). Therefore, A is a K-isomorphism between simply connected finite CW-complexes, but it is *not* a homotopy equivalence. The mapping cone C_A is a non-contractible finite CW-complex with $\widetilde{K}(C_A) = 0$. (It is non-contractible because its homology is non-trivial.)

iv) In [GrMo], pp. 203-206, a CW-complex $X = (S^1 \vee S^2) \cup e^3$ is defined, with the property that the inclusion $i\colon S^1 = X^{[1]} \hookrightarrow X$ of the 1-skeleton induces an isomorphism in integral homology (and on the level on fundamental groups); however, i is *not* a homotopy equivalence since $\pi_2(X) \neq 0$. Consequently, by the universal coefficient theorem (see Corollary V.7.2 in [Bred]), i induces an isomorphism in integral cohomology, and, by a direct application of the Atiyah-Hirzebruch spectral sequence, also in K-theory. In particular, i is a K-equivalence, but *not* an equivalence. (As C_A in the preceding example, the quotient space $X/X^{[1]}$ has vanishing $\widetilde{K}$, however it is the closed 3-ball and is therefore contractible.)

Let us finally mention that in [Matt], the positive cone, the c-cone and the γ-cone are also studied from the rational point of view, and rational K-theory is considered.

6. THE CONES OF THE PRODUCTS $S^n \times S^{2m-1}$

In this section, we will compute the cones for the products $S^{2n} \times S^{2m-1}$ and $S^{2n-1} \times S^{2m-1}$.

We begin with $S^{2n} \times S^{2m-1}$. Since $\widetilde{K}(S^{2m-1}) = 0$ and $K^1(S^{2n}) = 0$, the answer immediately follows from Proposition 5.5.

THEOREM 6.1. *The projection $p\colon S^{2n} \times S^{2m-1} \longrightarrow S^{2n}$ induces an isomorphism of positive cones, and, for $S^{2n} \times S^{2m-1}$, the γ-cone and the c-cone coincide with the positive cone:*

$$K_+(S^{2n}) \overset{p^*}{\cong} K_+(S^{2n} \times S^{2m-1}) = K_\gamma(S^{2n} \times S^{2m-1}).$$

We now turn to the product $S^{2n-1} \times S^{2m-1}$. From the six-term exact sequence of the pair $(S^{2n-1} \times S^{2m-1}, S^{2n-1} \vee S^{2m-1})$, with quotient the smash product $S^{2n-1} \wedge S^{2m-1}$ homeomorphic to $S^{2m+2n-2}$, we get an isomorphism

$$q^*: \widetilde{K}(S^{2m+2n-2}) \longrightarrow \widetilde{K}(S^{2n-1} \times S^{2m-1})$$

induced by the quotient map $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$. By Theorem 4.1, the space $Y = S^{2n+2m-2}$ satisfies the hypothesis of Proposition 5.5 and we deduce the

THEOREM 6.2. *The map $q: S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$ induces an isomorphism of positive cones, and, for $S^{2n-1} \times S^{2m-1}$, the γ-cone and the c-cone coincide with the positive cone:*

$$K_+(S^{2m+2n-2}) \overset{q^*}{\cong} K_+(S^{2n-1} \times S^{2m-1}) = K_\gamma(S^{2n-1} \times S^{2m-1}).$$

REMARK 6.3. According to Blackadar ([Bla2], 6.10.2), the positive cone of the n-torus $(S^1)^n$ has been partially computed by Villadsen.

7. THE γ-CONE OF $S^{2n} \times S^{2m}$ AND THE POSITIVE CONE OF $S^2 \times S^{2n}$

The positive cone was rather easy to compute for a product of an odd-dimensional sphere by any sphere, whereas the case of a product of two even-dimensional spheres is much more involved. On the other hand, the γ-cone of such a product is in the scope of the present notes. We perform this calculation by computing the c-cone and appealing to Proposition 3.3.

By the Künneth theorem, we have an isomorphism

$$K(S^{2n}) \otimes K(S^{2m}) \longrightarrow K(S^{2n} \times S^{2m}), \; \xi \otimes \eta \longmapsto p^*(\xi) \cdot q^*(\eta),$$

where p and q are the projections onto the factors. Writing $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x_1$ and $\widetilde{K}(S^{2m}) = \mathbf{Z} \cdot x_2$, and letting $y_1 := p^*(x_1)$ and $y_2 := q^*(x_2)$, we deduce that

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbf{Z} \cdot y_1 \oplus \mathbf{Z} \cdot y_2 \oplus \mathbf{Z} \cdot y_1 y_2 .$$

The product structure on $\widetilde{K}(S^{2n} \times S^{2m})$ is given by $y_1^2 = 0$ and $y_2^2 = 0$. One has $y_1 y_2 = \pi^*(y)$, where $\pi: S^{2n} \times S^{2m} \longrightarrow S^{2n} \wedge S^{2m} \cong S^{2n+2m}$ and y is a suitable generator of $\widetilde{K}(S^{2n+2m})$. Let $i: S^{2n} \hookrightarrow S^{2n} \times S^{2m}$ and $j: S^{2m} \hookrightarrow S^{2n} \times S^{2m}$ be the inclusions. One has $i^*(y_1) = x_1$ and $j^*(y_2) = x_2$, and (by Theorem 4.1 and a double application of Proposition 5.1), for any $k \in \mathbf{Z} \setminus \{0\}$, one has