10. "Gaps in cohomology" and the -cone

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REMARK 9.6.

i) This shows that $[x_1, x_2]$ generates $\pi_{11}(BU(3)) \cong \mathbf{Z}/30$ and that the map $j_*: \pi_{11}(BU(3)) \longrightarrow \pi_{11}(BU(5))$ is injective.

ii) We were also able to prove this theorem without appealing to results on homotopy groups of $BU(n)$. Using spectral sequence arguments, we have computed the first few stages of the Moore-Postnikov tower of the map $BSU(3) \longrightarrow BSU(5)$. This computation, being extremely lengthy, is not given here (see [Matt]).

Now we move on to the product $S^6 \times S^8$.

THEOREM 9.7. *For the product $S^6 \times S^8$, one has*

$$K_+(S^6 \times S^8) = K_c(S^6 \times S^8) = K_\gamma(S^6 \times S^8).$$

The latter is described in Theorem 7.1.

Proof. By Lundell's tables [Lun] (see also [Mim]), one has

$$\pi_{13}(BU(4)) \cong \mathbf{Z}/60 \quad \text{and} \quad \pi_{13}(BU(6)) \cong \mathbf{Z}/720.$$

Corollary 8.3 shows that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(6))$. By naturality, the map $j_* = \pi_{13}(j)$, induced by $j: BU(4) \longrightarrow BU(6)$, takes the Whitehead product $[x_1, x_2] \in \pi_{13}(BU(4))$ to $[x_1, x_2] \in \pi_{13}(BU(6))$. This implies that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(4))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_{13}(BU(4))$ precisely when it is zero in $\pi_{13}(BU(6))$. Together with Theorem 8.2, this completes the proof. $\square$

REMARK 9.8. The proof shows that $[x_1, x_2]$ is a generator of the group $\pi_{13}(BU(4)) \cong \mathbf{Z}/60$ and that the map $j_*: \pi_{13}(BU(4)) \longrightarrow \pi_{13}(BU(6))$ is injective.

10. "GAPS IN COHOMOLOGY" AND THE γ-CONE

In the present section, we are interested in spaces having a "gap in cohomology", more precisely we look at spaces obtained by attaching a single large-dimensional cell to a finite CW-complex Y. For such spaces, the integral cohomology is zero between the dimension of Y and the top-dimensional class. The products $S^n \times S^m$ are typical examples (see Section 8). For this kind of spaces, the c-cone obviously cannot give information in the dimensions corresponding to the gap. At first sight, one could think that the γ-cone is more powerful in this range. Unfortunately, this is not the case: we show that the γ-cone (or equivalently the γ-dimension function) is also "blind" in some sense. Here is the precise statement.

PROPOSITION 10.1. *Let Y be a connected finite CW-complex of dimension $\leq 2n$, and let $X = C_f = Y \cup_f e^{2n+2m}$ be the mapping cone of a map $f\colon S^{2n+2m-1} \longrightarrow Y$, with $m \geq 1$. Then, for $x \in \widetilde{K}(X)$, one has*

$$\gamma^{n+m}(x) = 0 \implies \gamma^{n+l}(x) = 0 \quad \text{for all} \quad l = 1, \ldots, m\,.$$

In other words, if $\gamma\text{-dim}(x) < n+m$, then $\gamma\text{-dim}(x) \leq n$.

Proof. By assumption, one has $H^k(X; \mathbf{Z}) = 0$ for $2n < k < 2n+2m$ and $H^{2n+2m}(X; \mathbf{Z}) \cong \mathbf{Z}$. Let $x \in \widetilde{K}(X)$ such that $\gamma^{n+m}(x) = 0$. By Proposition 2.2, keeping the same notation, we have

$$ch(\gamma^k(x)) = \bar{c}_k(x) + P_{k+1}(\bar{c}_1(x), \ldots, \bar{c}_{n+m}(x))\,,$$

and $0 = ch(\gamma^{n+m}(x)) = \bar{c}_{n+m}(x)$. Due to the "gap" in the cohomology of X, we find that, for $k > n$, we have

$$ch(\gamma^k(x)) = 0\,.$$

By the particular cohomological properties of X, the Chern character is injective for elements of filtration $> n$ in $\widetilde{K}(X)$ (see [AtHi]). Being zero or of filtration $\geq k$ (as Proposition 2.2 shows), $\gamma^k(x)$ has to vanish for $k > n$. This concludes the proof. □

11. A "DOUBLING FORMULA" FOR STIRLING NUMBERS OF THE SECOND KIND

In the present section, we calculate the γ-operations for the product $S^{2n} \times S^{2m}$. From this computation and Proposition 10.1, we deduce again the γ-cone, as appearing in Theorem 7.1. This example illustrates that computing the c-cone is in general easier than computing the γ-cone. On the other hand, the latter calculation leads to an interesting "doubling formula" for Stirling numbers of the second kind. We will also conjecture the analogous formula for Stirling numbers of the first kind.