2. Nilpotent Lie algebras with a unique rational form up to isomorphism

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

THEOREM 2. *Let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 2$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (regarded over $\mathbf{R}$) have infinitely many non-isomorphic rational forms.*

In Theorem 3 we also classify all rational forms for three 6-dimensional real nilpotent Lie algebras $\mathfrak{g}$ (two of them appear in Theorem 2 for $p = c = 2$) which are of class 2 and have 2-dimensional centre coinciding with the derived subalgebra.

In conclusion let us mention a direct way to prove that two given lattices in a nilpotent Lie group are not commensurable. For example, let $G = UT_3(\mathbf{R})$ be the Lie group of upper triangular 3×3-matrices with 1 on the diagonal, $\mathfrak{g} = \mathfrak{f}_2(2, \mathbf{R})$ be Lie algebra of G. Consider $G \times G$ and its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ which has infinitely many non-isomorphic rational forms $\mathfrak{h}_m$ ($m \geq 1$ is a square-free integer), in view and in the notation of Theorems 2, 3 (see Section 4 for more details).

Let Γ_m and Γ_n be corresponding lattices in $G \times G$ for distinct m, n. One can prove that the ratio of the covolumes of Γ_m and Γ_n with respect to a Haar measure on $G \times G$ equals $m\sqrt{m}/n\sqrt{n}$ up to a rational factor. Hence the lattices are not commensurable. Note that by Proposition 1.1 and Theorem 3 Γ_m and Γ_n are not commensurable in any sense.

ACKNOWLEDGEMENTS. The author is grateful to Pierre de la Harpe who attracted his attention to problems of commensurability and to the University of Geneva for its hospitality. He thanks also Goulnara Arzhantseva and Thierry Vust for helpful remarks.

2. NILPOTENT LIE ALGEBRAS WITH A UNIQUE RATIONAL FORM UP TO ISOMORPHISM

2.1 HEISENBERG ALGEBRAS

Let us begin with the following considerations that we will use here and in the next sections (see [2, Chapter 5] for more details). Suppose that a real Lie algebra $\mathfrak{g}$ has a $\mathbf{Q}$-form $\mathfrak{h}$ and $\mathfrak{i}$ (resp. $\mathfrak{a}$) is an ideal (resp. a subalgebra) of $\mathfrak{g}$. We say that $\mathfrak{i}$ (resp. $\mathfrak{a}$) is rational if $\mathfrak{i} \cap \mathfrak{h}$ (resp. $\mathfrak{a} \cap \mathfrak{h}$) is a rational form of $\mathfrak{i}$ (resp. $\mathfrak{a}$). For instance, the terms $C^k\mathfrak{g}$ of the lower central series of $\mathfrak{g}$ are rational as well as centralizers of rational subalgebras or ideals. It is not hard to see that $\mathfrak{h}/\mathfrak{i} \cap \mathfrak{h}$ is a rational form of the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$.

Let

$$\mathfrak{g} = \mathfrak{i}_1 > \mathfrak{i}_2 > \cdots > \mathfrak{i}_{k+1} = 0 \tag{2.1}$$

be a descending series of rational ideals of $\mathfrak{g}$. We say that a basis $X = \{x_1, \ldots, x_d\}$ of a rational form $\mathfrak{h}$ is based on (2.1) if $x_1, \ldots, x_{p_1}$ generate $\mathfrak{g} \bmod \mathfrak{i}_2$, $x_1, \ldots, x_{p_2}$ generate $\mathfrak{g} \bmod \mathfrak{i}_3$ and so on. It can be shown that such a basis exists for any series (2.1). In the sequel we will use these kinds of bases for a suitable descending series dealing, for instance, with Heisenberg algebras.

Recall that the (generalized) Heisenberg algebra $\mathfrak{hei}_{2k+1}(\mathbf{R})$ has an $\mathbf{R}$-basis $H_1, \ldots, H_{2k+1}$ in which

$$[H_1, H_2] = [H_3, H_4] = \cdots = [H_{2k-1}, H_{2k}] = H_{2k+1}, \tag{2.2}$$

other brackets being trivial. Here the 1-dimensional centre is spanned by H_{2k+1}.

Given an extension (1.1) one can attach to it a 2-cocycle $\omega\colon \Lambda^2\mathfrak{a} \to \mathbf{R}$ in the usual way. Also ω can be regarded as a symplectic form on $\mathfrak{a}$. If $\mathfrak{b} = \mathfrak{hei}_{2k+1}(\mathbf{R})$ then ω is the canonical non-degenerate symplectic form with respect to the basis $H_1, \ldots, H_{2k}$ (mod $\mathbf{R} \cdot H_{2k+1}$).

Let $d = \dim_{\mathbf{R}} \mathfrak{a}$ and let $m = d - rank(\omega)$ be the codimension of the kernel of ω. It is not hard to see (cf. the proof of the proposition below) that the Lie algebra $\mathfrak{b}$ is uniquely defined up to $\mathbf{R}$-isomorphism by d and m. Namely,

$$\mathfrak{b} \cong \mathfrak{hei}_{d+1-m}(\mathbf{R}) \oplus \mathbf{R}^m .$$

This implies that the centre of $\mathfrak{b}$ is $(m+1)$-dimensional. Thus, two Lie algebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$ ($\dim_{\mathbf{R}} \mathfrak{b}_1 = \dim_{\mathbf{R}} \mathfrak{b}_2$) of type (1.1) are not isomorphic if $m_1 \neq m_2$.

Evidently, $\mathfrak{hei}_{2k+1}(\mathbf{Q})$ is a rational form for $\mathfrak{hei}_{2k+1}(\mathbf{R})$.

The following proposition holds.

PROPOSITION 2.1. *In the above notation let $\mathfrak{h}$ be a rational form of $\mathfrak{b}$. Let $d = \dim(\mathfrak{b}) - 1$, $m = \dim[\mathfrak{b}, \mathfrak{b}]$ and let $\mathbf{Q}^m$ denote the abelian Lie $\mathbf{Q}$-algebra of dimension m. Then*

$$\mathfrak{h} \cong \mathfrak{hei}_{d+1-m}(\mathbf{Q}) \oplus \mathbf{Q}^m$$

over $\mathbf{Q}$, i.e., there is a unique rational form for $\mathfrak{b}$ up to isomorphism.

Proof. Choose a $\mathbf{Q}$-basis $B_1, \ldots, B_{d+1}$ for $\mathfrak{h}$. Either all brackets $[B_i, B_j] = 0$, and then $\mathfrak{h} \cong \mathbf{Q}^{d+1}$, or there are i,j such that $[B_i, B_j] = C \neq 0$.

We may suppose that $C = B_{d+1}$. Thus the derived subalgebra of $\mathfrak{h}$ is spanned by B_{d+1}. The corresponding symplectic form ω is represented by a skew-symmetric $d \times d$ matrix $M = (\mu_{ij})$ with respect to the basis $B_1, \dots, B_d$ (mod $[\mathfrak{h}, \mathfrak{h}]$). Namely, $[B_i, B_j] = \mu_{ij} B_{d+1}$. Over $\mathbf{Q}$ one can choose a canonical symplectic basis $\widehat{B_1}, \dots, \widehat{B_d}$ (mod $[\mathfrak{h}, \mathfrak{h}]$) so that the matrix $\widehat{M}$ representing ω has l blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of ω is equal to $2l$ and $2l = d - m$. In the basis $B_1, \dots, B_{d+1}$ (we omit the 'hats') of $\mathfrak{h}$

$$[B_1, B_2] = [B_3, B_4] = \cdots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

2.2 Example of a free nilpotent algebra

Let $\mathfrak{f}_c(n, \mathbf{R})$ be the free nilpotent Lie algebra of class c on n generators. Then $\mathfrak{f}_c(n, \mathbf{R})$ has a unique rational form $\mathfrak{f}_c(n, \mathbf{Q})$ up to isomorphism (cf. Theorem 2).

Indeed, let $\mathfrak{h} = \langle x_1, \dots, x_n, \dots \rangle$ be a rational form of $\mathfrak{f}_c(n, \mathbf{R})$. We may suppose that $x_1, \dots, x_n$ span (modulo the derived subalgebra) $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$. Consequently, $\mathfrak{h}$ is generated by $\{x_1, \dots, x_n\}$ as a Lie algebra. There exists an epimorphism $\pi \colon \mathfrak{f}_c(n, \mathbf{Q}) \to \mathfrak{h}$ because $\mathfrak{f}_c(n, \mathbf{Q})$ is free. It must be an isomorphism since the dimension of $\mathfrak{h}$ equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class c on n generators.

2.3 More examples

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let $\mathfrak{g}_t$, $t \in \mathbf{R}$, be a family of real 6-dimensional Lie algebras with a basis $\{x_1, \dots, x_6\}$ such that

$$[x_1, x_2] = x_3\,, \quad [x_1, x_3] = t x_5\,, \quad [x_1, x_5] = x_6\,,$$
$$[x_2, x_3] = x_4\,, \quad [x_2, x_4] = x_5\,, \quad [x_3, x_4] = x_6\,,$$

other brackets being trivial. One can show that

1. $C^k \mathfrak{g}_t = \langle x_{k+1}, \dots, x_6 \rangle$, $k = 2, \dots, 5$, where $C^k \mathfrak{g}_t$ are the terms of the lower central series of $\mathfrak{g}_t$.
2. The centralizer $\mathfrak{C}$ of $C^4 \mathfrak{g}_t$, that is, $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$ is spanned by $x_2, \dots, x_6$.

3. Real Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are not isomorphic but $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$.
4. If $t \in \mathbf{Q} \setminus \{0\}$ then the rational algebra $\mathfrak{g}_t \cong \mathfrak{g}_1$ over $\mathbf{Q}$.
5. $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are two Lie algebras with a unique rational form up to isomorphism.
6. Let $\mathfrak{g}$ be a split real simple Lie algebra of type G_2, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then $\mathfrak{n}_+$ is isomorphic to $\mathfrak{g}_0$.

3. Malcev's example

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a $\mathbf{Q}$-isomorphism between $\mathfrak{g}_t$ and $\mathfrak{g}_s$. It must be written in the following form (cf. [5]) since $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$, $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$ and the centralizer $\mathfrak{c}$ of $C^2\mathfrak{g}$, which is an ideal in this case, is spanned by $x_3, \dots, x_6$.

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots \\ y_3 = a_{33}x_3 + a_{34}x_4 + \dots \\ y_4 = a_{44}x_4 + \dots \end{cases}$$

We do not explicit the expressions for y_5, y_6. Here $y_1, \dots, y_6$ are basis elements of $\mathfrak{g}_s$ satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \dots ,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$. On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

$$\begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases} \tag{3.1}$$

We need to compute the remaining two brackets. First of all,

$$\begin{aligned} [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6. \end{aligned} \tag{3.2}$$