4.2 Proof of Theorem 2

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4. Nilpotent Lie algebras with infinitely many non-isomorphic rational forms

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

4.1 Basic lemma

Let

$$\mathfrak{h} = \bigoplus_{i=1}^{c} \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over $\mathbf{Q}$ generated by $\mathfrak{h}_1$. Let $\mathbf{K}$ be a number field, $\dim_{\mathbf{Q}} \mathbf{K} = d$, of type (s,t), that is, there are s real and $2t$ complex embeddings of $\mathbf{K}$ in $\mathbf{C}$ $(d = s+2t)$ whence there exists an isomorphism of $\mathbf{R}$-algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^{s} \mathbf{R} \oplus \bigoplus_{l=1}^{t} \mathbf{C}\,.$$

More generally one can take a finite-dimensional commutative associative algebra $\mathbf{A}$ over $\mathbf{Q}$ instead of $\mathbf{K}$. We consider the Lie algebra $\mathfrak{h}(\mathbf{K}) = \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}$ as a Lie algebra over $\mathbf{Q}$. This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^{s} \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^{t} \mathfrak{h}(\mathbf{C})\,,$$

i.e., $\mathfrak{h}(\mathbf{K})$ is a $\mathbf{Q}$-form of the last Lie algebra for any number field $\mathbf{K}$ of type (s,t). Secondly, there is an embedding $R\colon \mathbf{K}^* \to \mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$ of the multiplicative group $\mathbf{K}^*$ such that $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$ where $h_i \in \mathfrak{h}_i$ is homogenenous of degree i. The following lemma is straightforward.

Lemma 4.1. *Let $\mathbf{K} \neq \mathbf{K}'$ be two distinct number fields of the same type. If there is no injection of $\mathbf{K}^*$ into $\mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$ then two $\mathbf{Q}$-forms $\mathfrak{h}(\mathbf{K})$ and $\mathfrak{h}(\mathbf{K}')$ are not isomorphic.*

4.2 Proof of Theorem 2

We start with the class of nilpotence $c = 2$. Let $\mathbf{K} = \mathbf{Q}(\sqrt{m})$ and $\mathbf{K}' = \mathbf{Q}(\sqrt{n})$, where $m \neq n$ are two positive (resp. negative) square-free integers. Consider the automorphism $A = R(\sqrt{m})$ of $\mathfrak{h}(\mathbf{K}) = \mathfrak{f}_2(p,\mathbf{K})$. One immediately checks that

1) A^2 acts on $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}),\mathfrak{h}(\mathbf{K})]$ as $m \cdot Id$;

2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}),\mathfrak{h}(\mathbf{K})]} = m \cdot Id\,.$$

By Lemma 4.1 we must prove that there is no such automorphism for $\mathfrak{h}(\mathbf{K}') = \mathfrak{h}(\mathbf{Q}(\sqrt{n}))$. We choose the following basis of $\mathfrak{h}(\mathbf{K}')$ over $\mathbf{Q}$:

$$X_i = x_i \otimes 1, \quad Y_i = x_i \otimes \sqrt{n}, \quad C_{ij} = c_{ij} \otimes 1, \quad Z_{ij} = c_{ij} \otimes \sqrt{n},$$

$x_1, \dots, x_p, c_{ij} = [x_i, x_j]$ being the standard basis of $\mathfrak{f}_2(p, \mathbf{Q})$.

Suppose that there exists an automorphism A' with two above properties. First of all, let us show that $[X_i, A'(X_i)] = 0$. On the one hand,

$$A'[X_i, A'(X_i)] = [A'(X_i), mX_i] = -m[X_i, A'(X_i)].$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_i)].$$

Since the centralizer of X_i is generated modulo the centre by X_i, Y_i it follows that

$$A'(X_i) = p_i X_i + q_i Y_i + \varepsilon = x_i \otimes (p_i + q_i\sqrt{n}) + \varepsilon, \quad q_i \neq 0.$$

Here ε stands for a central element which plays no role below.

Consider now $[X_i, A'(X_j)] = c_{ij} \otimes (p_j + q_j\sqrt{n})$. On the one hand,

$$A'[X_i, A'(X_j)] = [A'(X_i), mX_j] = c_{ij} \otimes m(p_i + q_i\sqrt{n}).$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_j)] = c_{ij} \otimes m(p_j + q_j\sqrt{n}),$$

whence

$$p_i + q_i\sqrt{n} = p_j + q_j\sqrt{n} = p + q\sqrt{n} \notin \mathbf{Q} \ \forall i, j.$$

Finally, we apply A' to $[A'(X_i), A'(X_j)] = c_{ij} \otimes (p + q\sqrt{n})^2$. On the one hand,

$$A'[A'(X_i), A'(X_j)] = [mX_i, mX_j] = c_{ij} \otimes m^2.$$

On the other hand,

$$A'[A'(X_i), A'(X_i)] = m[A'(X_i), A'(X_j)] = c_{ij} \otimes m(p + q\sqrt{n})^2.$$

It follows that $m = (p + q\sqrt{n})^2$. We have obtained a contradiction since $q \neq 0$. Thus, there are infinitely many non-isomorphic rational forms of $\mathfrak{f}_2(p, \mathbf{R}) \oplus \mathfrak{f}_2(p, \mathbf{R})$ and of $\mathfrak{f}_2(p, \mathbf{C})$.

More generally let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 3$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (as a Lie algebra over $\mathbf{R}$) also have infinitely many non-isomorphic rational forms. Consider the automorphism A as above and note that it respects the descending central series. Any isomorphism between $\mathfrak{f}_c(p, \mathbf{K})$ and $\mathfrak{f}_c(p, \mathbf{K}')$ must respect it, too. Then we can take the free nilpotent quotients of class 2 of both algebras and obtain a contradiction just like in the first part of the proof. $\square$

Thus, the case of a free nilpotent Lie algebra $\mathfrak{f}_c(p, \mathbf{C})$ (as a Lie algebra over $\mathbf{R}$) on p generators differs from the case 2.2.

REMARK. All rational forms of $\mathfrak{f}_2(2, \mathbf{C}) = \mathfrak{hei}_3(\mathbf{C})$ and $\mathfrak{f}_2(2, \mathbf{R}) \oplus \mathfrak{f}_2(2, \mathbf{R}) = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ are listed in Theorem 3.

COROLLARY 4.2. *There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$ where $F_c(p, \mathbf{R})$ is the free nilpotent Lie group on p free generators.*

4.3 CLASSIFICATION OF RATIONAL FORMS FOR SOME 6-DIMENSIONAL LIE ALGEBRAS

Let m be a rational number and $A_m = \mathbf{Q}[x]/(x^2 - m)$. A_m is a 2-dimensional commutative algebra over $\mathbf{Q}$ which depends only on m modulo square factors. Thus there are four types of A_m :

1) if $m = 1$ then $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$;
2) if $m > 1$ is a positive square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is a real quadratic field over $\mathbf{Q}$;
3) if $m = 0$ then A_0 is the algebra of dual numbers over $\mathbf{Q}$;
4) if m is a negative square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is an imaginary quadratic field over $\mathbf{Q}$.

Let $\mathfrak{hei}_3(A_m)$ be a Heisenberg algebra over A_m considered over $\mathbf{Q}$. Then $\mathfrak{hei}_3(A_m)$ is a rational form of either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$. More precisely,

THEOREM 3. *Let $\mathfrak{h}$ be a 6-dimensional nilpotent Lie algebra of class 2 over $\mathbf{Q}$. Suppose that $[\mathfrak{h}, \mathfrak{h}]$ coincides with the 2-dimensional centre of $\mathfrak{h}$. Then $\mathfrak{h} \cong \mathfrak{hei}_3(A_m)$ for some $m \in \mathbf{Q}$ as above.*

Moreover,

1) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R}) = \mathfrak{g}_+$ *iff* $m > 0$,
2) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$ *iff* $m = 0$,
3) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{C}) = \mathfrak{g}_-$ *iff* $m < 0$,

and up to isomorphism there are no more rational forms for $\mathfrak{g}_-$, $\mathfrak{g}_0$, $\mathfrak{g}_+$. The Lie algebras $\mathfrak{hei}_3(A_m)$ and $\mathfrak{hei}_3(A_n)$ are isomorphic over $\mathbf{Q}$ if and only if A_m and A_n are isomorphic.