

6.5 The Urysohn metric space

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

EXERCISE 11. Prove that the action of S_∞ on LO is continuous and minimal (that is, the orbit of each linear order is everywhere dense in LO).

Recall that a linear order \prec is called *dense* if it has no gaps. A dense linear order without least and greatest elements is said to be of type η . The collection LO_η of all linear orders of type η on \mathbf{Z} can be identified with the factor space $S_\infty / \text{Aut}(\prec)$ through the correspondence $\sigma \mapsto \sigma \prec$. Here \prec is some chosen linear order of type η on \mathbf{Z} and $\text{Aut}(\prec)$ stands for the group of order-preserving self-bijections of (\mathbf{Z}, \prec) , acting on the space of orders in a natural way: $(x \sigma \prec y) \Leftrightarrow \sigma^{-1}x \prec \sigma^{-1}y$.

EXERCISE 12. Show that under the above identification the uniform structure on LO_η , induced from the compact space LO, is the finest uniform structure making the quotient map $S_\infty \rightarrow S_\infty / \text{Aut}(\prec) \cong LO_\eta$ right uniformly continuous.

Let now X be a compact S_∞ -space. The topological subgroup $\text{Aut}(\prec)$ of S_∞ has a fixed point in X , say x' (Exercise 10). The mapping $S_\infty \ni \sigma \mapsto \sigma(x') \in X$ is constant on the left $\text{Aut}(\prec)$ -cosets and thus gives rise to a mapping $\varphi: LO_\eta \rightarrow X$. Using Exercise 12, it is easy to see that φ is right uniformly continuous and thus extends to a morphism of S_∞ -spaces $LO \rightarrow X$. We have established the following result.

THEOREM 6 (Glasner and Weiss [Gl-W]). *The compact space LO forms the universal minimal S_∞ -space.*

6.5 THE URYSOHN METRIC SPACE

The *universal Urysohn metric space* \mathbf{U} [Ur] is determined uniquely (up to an isometry) by the following conditions:

- (i) \mathbf{U} is a complete separable metric space;
- (ii) \mathbf{U} is ω -homogeneous, that is, every isometry between two finite subspaces of \mathbf{U} extends to an isometry of \mathbf{U} ;
- (iii) \mathbf{U} contains an isometric copy of every separable metric space.

A probabilistic description of this space was given by Vershik [Ver]: the completion of the space of integers equipped with a ‘sufficiently random’ metric is almost surely isometric to \mathbf{U} .

The group of isometries $\text{Iso}(\mathbf{U})$ with the compact-open topology is a Polish (complete metric separable) topological group, which also possesses

a universality property: it contains an isomorphic copy of every separable metric group [Usp]. See also [Gr3].

Using concentration of measure, one can prove that the group $\text{Iso}(\mathbf{U})$ is extremely amenable. The Ramsey–Dvoretzky–Milman property leads to the following Ramsey-type result:

Let F be a finite metric space, and let all isometric embeddings of F into \mathbf{U} be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbf{U}$ of G such that all isometric embeddings of F into \mathbf{U} that factor through G are monochromatic to within ε .

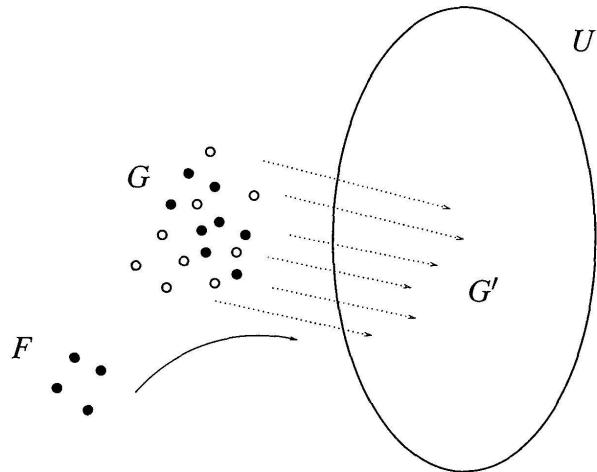


FIGURE 5

A Ramsey-type result for metric spaces

Here we say that a set A is *monochromatic to within ε* if there is a monochromatic set A' at a Hausdorff distance $< \varepsilon$ from A . In our case, the Hausdorff distance is formed with regard to the uniform metric on \mathbf{U}^F .

One can also obtain similar results, for example, for the separable Hilbert space ℓ_2 and for the unit sphere S^∞ in ℓ_2 [P3].

7. CONCENTRATION TO A NON-TRIVIAL SPACE

Let f be a Borel measurable real-valued function on an mm -space $X = (X, d, \mu)$. A number $M = M_f$ is called a *median* (or *Lévy mean*) of f if both $f^{-1}[M, +\infty)$ and $f^{-1}(-\infty, M]$ have measure $\geq \frac{1}{2}$.

EXERCISE 13. Show that the median M_f always exists, though it need not be unique.