4.1 Proof of Theorem 2.2

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4. Proof of Theorem 2.2 and Theorem 2.4

From now on, X stands for an eight-dimensional E-manifold with $w_2(X) = 0$.

4.1 Proof of Theorem 2.2

The classification result for 3-connected E-manifolds of dimension eight is a special case of a result of Wall's [36] and can be easily obtained with the methods described in [17], VII, § 12. Let us recall the details, because we will need them later on.

We fix a basis $\underline{b}$ for $H_4(X, \mathbf{Z})$ and let $\underline{y}$ be the dual basis of $H^4(X, Z)$. Then there is a handle presentation $X = D^8 \cup H_1^4 \cup \cdots \cup H_{b'}^4 \cup D^8$ with $\underline{b}$ as the preferred basis. The manifold $T := D^8 \cup H_1^4 \cup \cdots \cup H_{b'}^4$ is determined by the ambient isotopy class of a framed link of 3-spheres in S^7, having b' components. Let us first look at such a link, forgetting the framing, i.e., suppose we are given embeddings $g_i \colon S^3 \longrightarrow S^7$ with $S_i \cap S_j = \varnothing$ for $i \neq j$, $S_i := g_i(S^3)$, $i = 1, \dots, b'$. By 3.9, we may assume that the g_i are differentiable. Observe that the normal bundles of the S_i are trivial.

We equip S_i with the orientation induced via g_i by the standard orientation of S^3 and the normal bundle of S_i with the orientation which is determined by requiring that the orientation of S_i followed by that of its normal bundle coincide with the orientation of S^7. Therefore, a 3-sphere F_i which bounds the fibre of a tubular neighborhood of S_i in S^7 inherits an orientation and thus provides a generator e_i for $H_3(S^7 \setminus S_i, \mathbf{Z}) \cong \mathbf{Z}$, $i = 1, \dots, b'$. For $i \neq j$, the image of the fundamental class $[S_i]$ in $H_3(S^7 \setminus S_j, \mathbf{Z})$ is of the form $\lambda_{ij} \cdot e_j$. The integer λ_{ij} is called *the linking number of S_i and S_j*.

For $i = 1, \dots, b'$, the manifold $S^7 \setminus \bigcup_{j \neq i} S_j$ is up to dimension 5 homotopy equivalent to $\bigvee_{j \neq i} F_j$, and

$$\pi_3\big(S^7 \setminus \bigcup_{j \neq i} S_j\big) \cong \pi_3\big(\bigvee_{j \neq i} F_j\big) \cong \bigoplus_{j \neq i} H_3\big(S^7 \setminus S_j, \mathbf{Z}\big).$$

Under this identification, we have $[g_i] = \sum_{j \neq i} \lambda_{ij} \cdot e_j$. The $[g_i]$ determine the ambient isotopy class of the given link (3.9), and we deduce

PROPOSITION 4.1. *The linking numbers λ_{ij}, $1 \leq i < j \leq b'$, determine the given link up to ambient isotopy.*

The sphere S_i bounds a 4-dimensional disc D_i^- in D^8, $i = 1, \dots, b'$, which we equip with the induced orientation. We may, furthermore, assume that the D_i^- intersect transversely in the interior of D^8. Then the λ_{ij} coincide with the intersection numbers $D_i^-.D_j^-$, $1 \leq i < j \leq b'$. For an intuitive proof (in dimension 4), see [28], p. 67. Now, every disc D_i^- is completed by the core disc D_i^+ of the i^{th} 4-handle to an embedded 4-sphere Σ_i in T, $i = 1, \dots, b'$, and, since all the core discs are pairwise disjoint, the λ_{ij} coincide with the intersection numbers $\Sigma_i.\Sigma_j$, $1 \leq i < j \leq b'$. Finally, X is obtained by gluing an 8-disc to T along ∂T, and the spheres Σ_i represent the elements of the chosen basis $\underline{b}$ of $H_4(X, \mathbf{Z})$. Identifying the intersection ring with the cohomology ring of X via Poincaré-duality, we see

COROLLARY 4.2. *The linking numbers λ_{ij} coincide with the cup products $(y_i \cup y_j)[X]$, $1 \leq i < j \leq b'$, i.e., the link of the attaching spheres is determined up to ambient isotopy by the basis $\underline{b}$ and the cup products.*

As we have remarked before, the normal bundles of the S_i are trivial, whence there exist embeddings $f_i^0 \colon S^3 \times D^4 \longrightarrow S^7$ with $f^0|_{S^3 \times \{0\}} = g_i$, $i = 1, \dots, b'$. From the uniqueness of tubular (in differential topology) or regular (in piecewise linear topology) neighbourhoods, every other embedding $f_i \colon S^3 \times D^4 \longrightarrow S^7$ with $f_i|_{S^3 \times \{0\}} = g_i$ is ambient isotopic to one of the form $f_i^{[h_i]} := \big((x, y) \longmapsto (x, h_i \cdot y)\big)$, $[h_i] \in \pi_3(\mathrm{SO}(4))$, $i = 1, \dots, b'$. Corollary 3.14 implies that we can choose the f_i^0, $i = 1, \dots, b'$, in such a way that the following holds:

LEMMA 4.3. *Suppose T is obtained by attaching 4-handles along $f_i^{[h_i]}$ with $[h_i] = k_1^i \alpha_3 + k_2^i \beta_3$, $i = 1, \dots, b'$, then*

$$\Sigma_i.\Sigma_i = k_2^i \quad \textit{and} \quad p_1(T_{T|\Sigma_i}) = \pm\big(2k_2^i + 4k_1^i\big) .$$

This shows that also the framed link used for constructing T and X is determined by the system of invariants associated to $(X, \underline{y})$, proving the injectivity in Part i) of the theorem. Moreover, the assertion about the fibres in Part ii) is clear.

Conversely, given a system Z of invariants in $Z(0, b')$, satisfying relation (2), there exists a based 3-connected manifold $(X, \underline{y})$ realizing Z. Indeed, by the above identification of the invariants, Z determines a framed link in S^7 and thus the manifold $T := D^8 \cup H_1^4 \cup \cdots \cup H_{b'}^4$. The boundary of T is a 7-dimensional homotopy sphere ([17], (12.2), p. 119) and, therefore, piecewise linearly homeomorphic to S^7. Hence, $X = T \cup_{S^7} D^8$ is a piecewise linear manifold with the desired system of invariants, settling Part i). If, in addition, relation (3) holds, then [18] ensures that X will carry a smooth structure (compare Theorem A.4 of [24]), finishing the proof of Part ii). □

4.2 The determination of W_4 in the general case

We have a handle decomposition $W_0 \subset W_2 \subset W_4 \subset W_6 \subset X$ of X providing preferred bases $\underline{b}$ of $H_2(X,\mathbf{Z})$ and $\underline{c}$ of $H_4(X,\mathbf{Z})$, respectively. Let $\underline{x}$ and $\underline{y}$ be the dual bases of $H^2(X,\mathbf{Z})$ and $H^4(X,\mathbf{Z})$, respectively. Finally, let $\underline{y}^*$ be the basis of $H^4(X,\mathbf{Z})$ which is dual to $\underline{y}$ via γ_X.

We find $\partial W_2 \cong \#_{i=1}^b(S^2 \times S^5)$, and W_4 is determined by the ambient isotopy class of a framed link of 3-spheres in ∂W_2 with b' components. Let $f_k\colon S^3 \times D^4 \longrightarrow \partial W_2$ be the k^{th} component of that link and $g_k := f_k|_{S^3\times\{0\}}$, $k = 1,\ldots,b'$. In the notation of Section 3.6, we write $[g_k] \in \pi_3(\partial W_2 \setminus \bigcup_{k\neq j} S_j)$ in the form $(l_i^k, i = 1,\ldots,b, l_{ij}^k, 1 \le i < j \le b; \lambda_{kj}, j \neq k)$, $k = 1,\ldots,b'$. To see the significance of the l_i^k and l_{ij}^k, note that, by Remark 3.4, $W_2 \cup H_k^4 \subset X$ is homotopy equivalent to $\left(\bigvee_{i=1}^b S^2\right) \cup_{g_k} D^4$. The cohomology ring of that complex has been computed in Proposition 3.11, so that the naturality of the cup product implies the following formulae for the cup products in X:

$$x_i \cup x_j = \sum_{k=1}^{b'} l_{ij}^k \cdot y_k^*, \quad i \neq j,$$

$$x_i \cup x_i = \sum_{k=1}^{b'} l_i^k \cdot y_k^*, \quad i = 1,\ldots,b.$$

Therefore, the l_i^k and l_{ij}^k are determined by δ_X and γ_X (used to compute $\underline{y}^*$), in fact $l_i^k = \gamma_X(\delta(x_i \otimes x_i) \otimes y_k)$ and $l_{ij}^k = \gamma_X(\delta(x_i \otimes x_j) \otimes y_k)$.

To determine the λ_{ij} and the framings, we proceed as follows: Look at the embedding $\#_{i=1}^b(S^2 \times S^5) \hookrightarrow X$. There exist b embedded 2-spheres $S_1^2,\ldots,S_b^2$ which represent the basis $\underline{b}$ and which do not meet the given link. Finally, $\#_{i=1}^b(S^2 \times S^5)$ obviously possesses a regular neighborhood in X which is homeomorphic to $\#_{i=1}^b(S^2 \times S^5) \times D^1$. Thus, we can perform "surgery in pairs" as described in Section 3.1. The result is a 3-connected manifold X^* containing S^7. It is by construction the manifold obtained from the framed link in S^7 derived from the given one in $\#_{i=1}^b(S^2 \times S^5)$ (cf. Section 4.1). We will be finished, once we are able to compare the invariants of X to those of X^*. To do so, we look at the *trace of the surgery*, i.e., at $Y = (X \times I) \cup H_1^5 \cup \cdots \cup H_{b'}^5$, the 5-handles being attached along tubular neighborhoods of the $S_i \times \{1\}$ in $X \times \{1\}$. Then $\partial Y = X \sqcup \overline{X}^*$.