addition, relation (3) holds, then [18] ensures that X will carry a smooth structure (compare Theorem A.4 of [24]), finishing the proof of Part ii). $\square$

4.2 THE DETERMINATION OF W_4 IN THE GENERAL CASE

We have a handle decomposition $W_0 \subset W_2 \subset W_4 \subset W_6 \subset X$ of X providing preferred bases $\underline{b}$ of $H_2(X,\mathbf{Z})$ and $\underline{c}$ of $H_4(X,\mathbf{Z})$, respectively. Let $\underline{x}$ and $\underline{y}$ be the dual bases of $H^2(X,\mathbf{Z})$ and $H^4(X,\mathbf{Z})$, respectively. Finally, let $\underline{y}^*$ be the basis of $H^4(X,\mathbf{Z})$ which is dual to $\underline{y}$ via γ_X.

We find $\partial W_2 \cong \#_{i=1}^b(S^2 \times S^5)$, and W_4 is determined by the ambient isotopy class of a framed link of 3-spheres in ∂W_2 with b' components. Let $f_k \colon S^3 \times D^4 \longrightarrow \partial W_2$ be the k^{th} component of that link and $g_k := f_k|_{S^3 \times \{0\}}$, $k = 1,\ldots,b'$. In the notation of Section 3.6, we write $[g_k] \in \pi_3(\partial W_2 \setminus \bigcup_{k \neq j} S_j)$ in the form $(l_i^k, i = 1,\ldots,b, l_{ij}^k, 1 \le i < j \le b; \lambda_{kj}, j \neq k)$, $k = 1,\ldots,b'$. To see the significance of the l_i^k and l_{ij}^k, note that, by Remark 3.4, $W_2 \cup H_k^4 \subset X$ is homotopy equivalent to $\left(\bigvee_{i=1}^b S^2\right) \cup_{g_k} D^4$. The cohomology ring of that complex has been computed in Proposition 3.11, so that the naturality of the cup product implies the following formulae for the cup products in X:

$$x_i \cup x_j = \sum_{k=1}^{b'} l_{ij}^k \cdot y_k^*, \quad i \neq j,$$

$$x_i \cup x_i = \sum_{k=1}^{b'} l_i^k \cdot y_k^*, \quad i = 1,\ldots,b.$$

Therefore, the l_i^k and l_{ij}^k are determined by δ_X and γ_X (used to compute $\underline{y}^*$), in fact $l_i^k = \gamma_X(\delta(x_i \otimes x_i) \otimes y_k)$ and $l_{ij}^k = \gamma_X(\delta(x_i \otimes x_j) \otimes y_k)$.

To determine the λ_{ij} and the framings, we proceed as follows: Look at the embedding $\#_{i=1}^b(S^2 \times S^5) \hookrightarrow X$. There exist b embedded 2-spheres $S_1^2,\ldots,S_b^2$ which represent the basis $\underline{b}$ and which do not meet the given link. Finally, $\#_{i=1}^b(S^2 \times S^5)$ obviously possesses a regular neighborhood in X which is homeomorphic to $\#_{i=1}^b(S^2 \times S^5) \times D^1$. Thus, we can perform "surgery in pairs" as described in Section 3.1. The result is a 3-connected manifold X^* containing S^7. It is by construction the manifold obtained from the framed link in S^7 derived from the given one in $\#_{i=1}^b(S^2 \times S^5)$ (cf. Section 4.1). We will be finished, once we are able to compare the invariants of X to those of X^*. To do so, we look at the *trace of the surgery*, i.e., at $Y = (X \times I) \cup H_1^5 \cup \cdots \cup H_{b'}^5$, the 5-handles being attached along tubular neighborhoods of the $S_i \times \{1\}$ in $X \times \{1\}$. Then $\partial Y = X \sqcup \overline{X}^*$.

The Mayer-Vietoris sequence provides the isomorphisms

$$H_4(X, \mathbf{Z}) \cong H_4\big(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\big) \cong H_4(X^*, \mathbf{Z}).$$

Set $H := H_4\big(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\big)$. By Lefschetz duality ([5], (28.18)), there is for each $q \in \mathbf{N}$ a diagram (omitting $\mathbf{Z}$-coefficients)

(4)
$$\begin{tikzcd} H^{q-1}(Y) \arrow[r] \arrow[d, "\cong"] & H^{q-1}(\partial Y) \arrow[r] \arrow[d, "\cong"] & H^q(Y, \partial Y) \arrow[r] \arrow[d, "\cong"] & H^q(Y) \arrow[d, "\cong"] \\ H_{10-q}(Y, \partial Y) \arrow[r] & H_{9-q}(\partial Y) \arrow[r] & H_{9-q}(Y) \arrow[r] & H_{9-q}(Y, \partial Y) \end{tikzcd}$$

where the left square commutes up to the sign $(-1)^{q-1}$ and the other two commute. We first use it in the case $q = 5$. Look at the commutative diagram

$$\begin{tikzcd} H \arrow[r, "\cong"] \arrow[d, "\cong"] & H_4(X^*, \mathbf{Z}) \arrow[d] \\ H_4(X, \mathbf{Z}) \arrow[r] & H_4(Y, \mathbf{Z}), \end{tikzcd}$$

in which all arrows are injective, because $H_5(Y, X; \mathbf{Z}) = 0 = H_5(Y, X^*; \mathbf{Z})$ (cf. [17], p. 198). Using the identification $H_4(\partial Y, \mathbf{Z}) = H \oplus H$, we find

(5)
$$\operatorname{Im}\big(H_5(Y, \partial Y; \mathbf{Z})\big) = \{\, (y, -y) \in H \oplus H \,\}.$$

Similar considerations apply to the case $q = 9$. Taking into account that X^* sits in Y with the reversed orientation, (4) shows that the forms γ_X and γ_{X^*}, both defined with respect to the preferred bases, coincide. In the same manner, the pullbacks of $p_1(Y)$ to $H^4(X, \mathbf{Z})$ and $H^4(X^*, \mathbf{Z})$, respectively, agree. Since X and X^* are the boundary components of Y, these pullbacks are $p_1(X)$ and $p_1(X^*)$, respectively, and we are done. $\square$

4.3 Manifolds with given invariants

One might speculate, especially in view of the classification of E-manifolds in dimension 4 and 6, that the invariants δ_X, γ_X, and $p_1(X)$ might suffice to classify E-manifolds with $w_2(X) = 0$ in dimension 8. However,