

## **4.2 THE DETERMINATION OF $w_4$ IN THE GENERAL CASE**

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

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addition, relation (3) holds, then [18] ensures that  $X$  will carry a smooth structure (compare Theorem A.4 of [24]), finishing the proof of Part ii).  $\square$

#### 4.2 THE DETERMINATION OF $W_4$ IN THE GENERAL CASE

We have a handle decomposition  $W_0 \subset W_2 \subset W_4 \subset W_6 \subset X$  of  $X$  providing preferred bases  $\underline{b}$  of  $H_2(X, \mathbf{Z})$  and  $\underline{c}$  of  $H_4(X, \mathbf{Z})$ , respectively. Let  $\underline{x}$  and  $\underline{y}$  be the dual bases of  $H^2(X, \mathbf{Z})$  and  $H^4(X, \mathbf{Z})$ , respectively. Finally, let  $\underline{y}^*$  be the basis of  $H^4(X, \mathbf{Z})$  which is dual to  $\underline{y}$  via  $\gamma_X$ .

We find  $\partial W_2 \cong \#_{i=1}^b (S^2 \times S^5)$ , and  $W_4$  is determined by the ambient isotopy class of a framed link of 3-spheres in  $\partial W_2$  with  $b'$  components. Let  $f_k: S^3 \times D^4 \rightarrow \partial W_2$  be the  $k^{\text{th}}$  component of that link and  $g_k := f_k|_{S^3 \times \{0\}}$ ,  $k = 1, \dots, b'$ . In the notation of Section 3.6, we write  $[g_k] \in \pi_3(\partial W_2 \setminus \bigcup_{k \neq j} S_j)$  in the form  $(l_i^k, i = 1, \dots, b, l_{ij}^k, 1 \leq i < j \leq b; \lambda_{kj}, j \neq k)$ ,  $k = 1, \dots, b'$ . To see the significance of the  $l_i^k$  and  $l_{ij}^k$ , note that, by Remark 3.4,  $W_2 \cup H_k^4 \subset X$  is homotopy equivalent to  $(\bigvee_{i=1}^b S^2) \cup_{g_k} D^4$ . The cohomology ring of that complex has been computed in Proposition 3.11, so that the naturality of the cup product implies the following formulae for the cup products in  $X$ :

$$\begin{aligned} x_i \cup x_j &= \sum_{k=1}^{b'} l_{ij}^k \cdot y_k^*, \quad i \neq j, \\ x_i \cup x_i &= \sum_{k=1}^{b'} l_i^k \cdot y_k^*, \quad i = 1, \dots, b. \end{aligned}$$

Therefore, the  $l_i^k$  and  $l_{ij}^k$  are determined by  $\delta_X$  and  $\gamma_X$  (used to compute  $\underline{y}^*$ ), in fact  $l_i^k = \gamma_X(\delta(x_i \otimes x_i) \otimes y_k)$  and  $l_{ij}^k = \gamma_X(\delta(x_i \otimes x_j) \otimes y_k)$ .

To determine the  $\lambda_{ij}$  and the framings, we proceed as follows: Look at the embedding  $\#_{i=1}^b (S^2 \times S^5) \hookrightarrow X$ . There exist  $b$  embedded 2-spheres  $S_1^2, \dots, S_b^2$  which represent the basis  $\underline{b}$  and which do not meet the given link. Finally,  $\#_{i=1}^b (S^2 \times S^5)$  obviously possesses a regular neighborhood in  $X$  which is homeomorphic to  $\#_{i=1}^b (S^2 \times S^5) \times D^1$ . Thus, we can perform “surgery in pairs” as described in Section 3.1. The result is a 3-connected manifold  $X^*$  containing  $S^7$ . It is by construction the manifold obtained from the framed link in  $S^7$  derived from the given one in  $\#_{i=1}^b (S^2 \times S^5)$  (cf. Section 4.1). We will be finished, once we are able to compare the invariants of  $X$  to those of  $X^*$ . To do so, we look at the *trace of the surgery*, i.e., at  $Y = (X \times I) \cup H_1^5 \cup \dots \cup H_{b'}^5$ , the 5-handles being attached along tubular neighborhoods of the  $S_i \times \{1\}$  in  $X \times \{1\}$ . Then  $\partial Y = X \sqcup \bar{X}^*$ .

The Mayer-Vietoris sequence provides the isomorphisms

$$H_4(X, \mathbf{Z}) \cong H_4\left(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\right) \cong H_4(X^*, \mathbf{Z}).$$

Set  $H := H_4\left(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\right)$ . By Lefschetz duality ([5], (28.18)), there is for each  $q \in \mathbf{N}$  a diagram (omitting  $\mathbf{Z}$ -coefficients)

$$(4) \quad \begin{array}{ccccccc} H^{q-1}(Y) & \longrightarrow & H^{q-1}(\partial Y) & \longrightarrow & H^q(Y, \partial Y) & \longrightarrow & H^q(Y) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_{10-q}(Y, \partial Y) & \longrightarrow & H_{9-q}(\partial Y) & \longrightarrow & H_{9-q}(Y) & \longrightarrow & H_{9-q}(Y, \partial Y) \end{array}$$

where the left square commutes up to the sign  $(-1)^{q-1}$  and the other two commute. We first use it in the case  $q = 5$ . Look at the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\cong} & H_4(X^*, \mathbf{Z}) \\ \downarrow \cong & & \downarrow \\ H_4(X, \mathbf{Z}) & \longrightarrow & H_4(Y, \mathbf{Z}), \end{array}$$

in which all arrows are injective, because  $H_5(Y, X; \mathbf{Z}) = 0 = H_5(Y, X^*; \mathbf{Z})$  (cf. [17], p. 198). Using the identification  $H_4(\partial Y, \mathbf{Z}) = H \oplus H$ , we find

$$(5) \quad \text{Im}(H_5(Y, \partial Y; \mathbf{Z})) = \{ (y, -y) \in H \oplus H \}.$$

Similar considerations apply to the case  $q = 9$ . Taking into account that  $X^*$  sits in  $Y$  with the reversed orientation, (4) shows that the forms  $\gamma_X$  and  $\gamma_{X^*}$ , both defined with respect to the preferred bases, coincide. In the same manner, the pullbacks of  $p_1(Y)$  to  $H^4(X, \mathbf{Z})$  and  $H^4(X^*, \mathbf{Z})$ , respectively, agree. Since  $X$  and  $X^*$  are the boundary components of  $Y$ , these pullbacks are  $p_1(X)$  and  $p_1(X^*)$ , respectively, and we are done.  $\square$

#### 4.3 MANIFOLDS WITH GIVEN INVARIANTS

One might speculate, especially in view of the classification of E-manifolds in dimension 4 and 6, that the invariants  $\delta_X$ ,  $\gamma_X$ , and  $p_1(X)$  might suffice to classify E-manifolds with  $w_2(X) = 0$  in dimension 8. However,