

## 4.3 Manifolds with given invariants

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The Mayer-Vietoris sequence provides the isomorphisms

$$H_4(X, \mathbf{Z}) \cong H_4\left(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\right) \cong H_4(X^*, \mathbf{Z}).$$

Set  $H := H_4\left(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\right)$ . By Lefschetz duality ([5], (28.18)), there is for each  $q \in \mathbf{N}$  a diagram (omitting  $\mathbf{Z}$ -coefficients)

$$(4) \quad \begin{array}{ccccccc} H^{q-1}(Y) & \longrightarrow & H^{q-1}(\partial Y) & \longrightarrow & H^q(Y, \partial Y) & \longrightarrow & H^q(Y) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_{10-q}(Y, \partial Y) & \longrightarrow & H_{9-q}(\partial Y) & \longrightarrow & H_{9-q}(Y) & \longrightarrow & H_{9-q}(Y, \partial Y) \end{array}$$

where the left square commutes up to the sign  $(-1)^{q-1}$  and the other two commute. We first use it in the case  $q = 5$ . Look at the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\cong} & H_4(X^*, \mathbf{Z}) \\ \downarrow \cong & & \downarrow \\ H_4(X, \mathbf{Z}) & \longrightarrow & H_4(Y, \mathbf{Z}), \end{array}$$

in which all arrows are injective, because  $H_5(Y, X; \mathbf{Z}) = 0 = H_5(Y, X^*; \mathbf{Z})$  (cf. [17], p. 198). Using the identification  $H_4(\partial Y, \mathbf{Z}) = H \oplus H$ , we find

$$(5) \quad \text{Im}(H_5(Y, \partial Y; \mathbf{Z})) = \{ (y, -y) \in H \oplus H \}.$$

Similar considerations apply to the case  $q = 9$ . Taking into account that  $X^*$  sits in  $Y$  with the reversed orientation, (4) shows that the forms  $\gamma_X$  and  $\gamma_{X^*}$ , both defined with respect to the preferred bases, coincide. In the same manner, the pullbacks of  $p_1(Y)$  to  $H^4(X, \mathbf{Z})$  and  $H^4(X^*, \mathbf{Z})$ , respectively, agree. Since  $X$  and  $X^*$  are the boundary components of  $Y$ , these pullbacks are  $p_1(X)$  and  $p_1(X^*)$ , respectively, and we are done.  $\square$

### 4.3 MANIFOLDS WITH GIVEN INVARIANTS

One might speculate, especially in view of the classification of E-manifolds in dimension 4 and 6, that the invariants  $\delta_X$ ,  $\gamma_X$ , and  $p_1(X)$  might suffice to classify E-manifolds with  $w_2(X) = 0$  in dimension 8. However,

Lemma 3.6 shows that these invariants determine only  $W_4$  and we still have the choice of an isomorphism in gluing  $\#_{i=1}^b(S^2 \times S^5)$  to  $W_4$ , and different gluings may lead to different results. The following example, which was communicated to me by J.-C. Hausmann, illustrates this phenomenon.

EXAMPLE 4.4. One has  $\pi_5(\mathrm{SO}(3)) \cong \mathbf{Z}_2$  [32]. Therefore, there are two different  $S^2$ -bundles over  $S^6$ , call them  $X := S^6 \times S^2$  and  $X' := S^6 \tilde{\times} S^2$ . Obviously,  $X$  and  $X'$  are spin-manifolds with trivial invariants, but one computes  $\pi_5(X) \cong \mathbf{Z}_2$  and  $\pi_5(X') = \{0\}$ .

Fix  $b, b'$ , and a system  $Z$  of invariants in the image of the map  $Z^{\mathrm{PL}(C^\infty)}(b, b')$ . As we have seen,  $Z$  determines a certain manifold  $W_4$  whose boundary is diffeomorphic to  $\#_{i=1}^b(S^2 \times S^5)$  together with a basis  $\underline{b}$  for  $H_2(\partial W_4, \mathbf{Z})$ . Let  $\underline{b}_0$  be the natural basis for  $H_2(\#_{i=1}^b(S^2 \times S^5), \mathbf{Z})$ , and denote by  $\mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$  the set of piecewise linear (smooth) isomorphisms  $f: \#_{i=1}^b(S^2 \times S^5) \rightarrow \partial W_4$  with  $f_*(\underline{b}_0) = \underline{b}$ . Our results show that every based piecewise linear (smooth) manifold  $(X, \underline{x}, \underline{y})$  with system of invariants  $Z$  is piecewise linearly (smoothly) isomorphic to a manifold of the form

$$X(f) := \partial W_4 \cup_f \#_{i=1}^b(S^2 \times S^5) \quad \text{for some } f \in \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$$

with its given bases for  $H^2(X(f), \mathbf{Z})$  and  $H^4(X(f), \mathbf{Z})$ . Conversely, every manifold of the form  $X(f)$  is a piecewise linear (smooth) based E-manifold with invariants  $Z$ .

Now, suppose we are given  $f, f' \in \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$ , such that  $X(f)$  and  $X(f')$  are isomorphic as piecewise linear (smooth) based manifolds. We claim that we can find an isomorphism  $\varphi: X(f) \rightarrow X(f')$  with  $\varphi(W_4) = W_4$ . For this, look at the handle decomposition  $W_0 \subset W_2 \subset W_4$ . Since  $W_0$  is just an embedded 8-disc in  $X(f)$  and  $X(f')$ , respectively, we can choose  $\varphi$  with  $\varphi(W_0) = W_0$ . Let  $l \subset \partial W_0$  be the framed link for attaching the 2-handles. Then  $\varphi(l)$  and  $l$  are isotopic. Therefore, we can find a level preserving diffeomorphism  $\tilde{\psi}: [-1, 1] \times \partial W_0 \rightarrow [-1, 1] \times \partial W_0$  with  $\tilde{\psi}|_{\{\pm 1\} \times \partial W_0} = \mathrm{id}_{\partial W_0}$  and  $\tilde{\psi}|_{\{0\} \times \partial W_0}(\varphi(l)) = l$ . If we choose a tubular neighborhood ( $\cong [-1, 1] \times \partial W_0$ ) of  $\partial W_0$  in  $X(f')$ , we can use  $\tilde{\psi}$  to define an automorphism  $\psi: X(f') \rightarrow X(f')$  with  $\psi(\varphi(l)) = l$ . Thus,  $\psi \circ \varphi$  maps  $W_2$  onto  $W_2$ . A similar argument shows that we can achieve  $\varphi(W_4) = W_4$ .

Let  $\text{Aut}_0^{\text{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times D^6))$  be the group of piecewise linear (smooth) automorphisms  $g$  of  $\#_{i=1}^b(S^2 \times D^6)$  with  $H^2(g, \mathbf{Z}) = \text{id}$  and similarly define  $\text{Aut}_0^{\text{PL}(C^\infty)}(W_4)$ . Then we have just established

**PROPOSITION 4.5.** *The set of isomorphism classes of based piecewise linear (smooth) E-manifolds with invariants  $\mathbf{Z}$  is in bijection to the set of equivalence classes in  $\text{Iso}_0^{\text{PL}(C^\infty)}$  with respect to the equivalence relation coming from the group action*

$$\begin{aligned} \text{Aut}_0^{\text{PL}(C^\infty)}(W_4) \times \text{Aut}_0^{\text{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times D^6)) \times \text{Iso}_0^{\text{PL}(C^\infty)} &\longrightarrow \text{Iso}_0^{\text{PL}(C^\infty)} \\ (h, g, f) &\longmapsto h|_{\partial W_4} \circ f \circ g|_{\#_{i=1}^b(S^2 \times S^5)}^{-1}. \end{aligned}$$

We shall see in Lemma 5.1 that  $\text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times D^6))$  contains the commutator subgroup of  $\text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5))$ .

**COROLLARY 4.6.** *The set of isomorphism classes of based piecewise linear E-manifolds with  $b_2 = b$  and  $b_4 = 0$  is in bijection to the abelian group*

$$\text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5)) / \text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times D^6)).$$

I have been informed by experts that the structure of the groups  $\text{Aut}_0^{\text{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times S^5))$  and  $\text{Aut}_0^{\text{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times D^6))$  has not yet been determined and that this would be a rather difficult task. Therefore, we choose the viewpoint of framed links in order to finish our considerations. In Theorem 5.2, we will then use this viewpoint to compute the group  $\text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5)) / \text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times D^6))$ .

As above, let  $(X, \underline{x}, \underline{y})$  be a based piecewise linear (smooth) E-manifold with zero second Stiefel-Whitney class and system of invariants  $\mathbf{Z}_{(X, \underline{x}, \underline{y})} = (\delta, \gamma, p)$ . We have seen that we can find a framed link  $l_X$  of 2-spheres in  $X$  which represents the basis  $\underline{x}$  and perform surgery along this link in order to get a 3-connected piecewise linear (smooth) based manifold  $(X^*, \underline{y})$  together with a framed link  $l_{X^*}^*$  of 5-spheres in it. If  $(X', \underline{x}', \underline{y}', l_{X'})$  is another such object where  $(X', \underline{x}', \underline{y}')$  is isomorphic to  $(X, \underline{x}, \underline{y})$ , then clearly we can find an isomorphism  $\varphi: (X, \underline{x}, \underline{y}) \longrightarrow (X', \underline{x}', \underline{y}')$  with  $\varphi(l_X) = l_{X'}$ . Such an isomorphism  $\varphi$  yields, after surgery, an isomorphism  $\varphi^*: (X^*, \underline{y}) \longrightarrow (X'^*, \underline{y}')$  with  $\varphi^*(l_{X^*}^*) = l_{X'^*}^*$ . In particular, the manifold  $(X^*, \underline{y})$  is determined up to piecewise linear (smooth) isomorphism. We call it the *type* of  $(X, \underline{x}, \underline{y})$ . Note that this notion matters only in the smooth case, by Theorem 2.2.

To summarize, we have

**PROPOSITION 4.7.** *The set of isomorphism classes of based piecewise linear (smooth) E-manifolds of type  $(X^*, \underline{y})$  is in bijection to the set of equivalence classes of framed links of 5-spheres in  $X^*$  where two such links  $l$  and  $l'$  are considered equivalent, if there is a piecewise linear (smooth) automorphism  $\varphi^*: (X^*, \underline{y}) \rightarrow (X^*, \underline{y})$  with  $\varphi^*(l) = l'$ .*

**EXAMPLE 4.8.** The group  $\mathbb{Z}_2^{\oplus b}$  acts freely on the set of isotopy classes of framed links of  $b$  spheres of dimension 5 in  $X^*$  by altering the framings of the components. Note that the two possible framings of the trivial bundle on a 5-sphere are distinguished by the fact that one extends over  $D^6$  and the other does not. This property is preserved under piecewise linear homeomorphisms, so that we conclude that  $\mathbb{Z}_2^{\oplus b}$  acts also freely on the set of equivalence classes of framed links of  $b$  spheres of dimension 5 in  $X^*$ .

Note that this completes the classification of Spin-E-manifolds of dimension eight with second Betti number one.

Let us look at manifolds of type  $S^8$ . We claim that two framed links  $l$  and  $l'$  of 5-spheres are equivalent in the above sense, if and only if they are isotopic. Clearly, after replacing  $l$  and  $l'$  by isotopic links, we may assume that both of them are contained in the Southern hemisphere and that  $\varphi^*$  is the identity on the Northern hemisphere. Now, choose a representative  $\varphi^\dagger$  for the isotopy class of  $\varphi^{*-1}$  which is the identity on the Southern hemisphere. Then  $\varphi^\dagger \circ \varphi^*$  is isotopic to the identity and carries  $l$  into  $l'$ .

For differentiable manifolds, the operation  $X \mapsto X \# \Sigma$ ,  $\Sigma$  an exotic 8-sphere, establishes a bijection between the set of isomorphism classes of based smooth E-manifolds of type  $S^8$  and the set of isomorphism classes of based smooth E-manifolds of type  $\Sigma$ . We conclude

**COROLLARY 4.9.** i) *The set of isomorphism classes of based piecewise linear E-manifolds with  $b_2 = b$  and  $b_4 = 0$  is in bijection to the group  $\text{FL}_b = \text{L}_b \oplus \bigoplus_{i=1}^b \mathbb{Z}_2$ .*

ii) *The set of isomorphism classes of based smooth E-manifolds with  $b_2 = b$  and  $b_4 = 0$  is in bijection to the group  $\vartheta^8 \oplus \text{FL}_b$ .*

Finally, we have to deal with those manifolds for which the cup form  $\delta$  is trivial. Our investigations in Sections 3.6 and 4.2 show that the framed link of 3-spheres in  $\partial W_2$  can be chosen to be contained in a small disc.

In other words, a manifold  $X$  with  $\delta_X \equiv 0$  is piecewise linear (smoothly) isomorphic  $X^\dagger \# X^*$  where  $X^*$  is the type of  $X$  and  $b_4(X^\dagger) = 0$ . As our surgery arguments above reveal, an isomorphism between  $X^\dagger \# X^*$  and  $X'^\dagger \# X'^*$  can be chosen of the form  $\varphi^\dagger \# \varphi^*$  where  $\varphi^\dagger: X^\dagger \rightarrow X'^\dagger$  and  $\varphi^*: X^* \rightarrow X'^*$  are isomorphisms. Therefore, the set of isomorphism classes of based piecewise linear E-manifolds of type  $X^*$  with  $b_2 = b$  is in bijection to the set of isomorphism classes of based piecewise linear E-manifolds with  $b_2 = b$  and  $b_4 = 0$ . The same goes for differentiable manifolds of type  $X^*$ , if  $X^*$  is not diffeomorphic to  $X^* \# \Sigma$ ,  $\Sigma$  an exotic 8-sphere. Otherwise, we have to divide by the action of  $\vartheta^8$ . This observation together with Corollary 4.9 settles Theorem 2.4.  $\square$

## 5. STRUCTURE OF THE GROUP $\text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5)) / \text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times D^6))$

In this section we prove that  $\text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5)) / \text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times D^6))$  is an abelian group which is, moreover, isomorphic to the group  $\text{FL}_b$  defined before. This result should be of some independent interest, especially because the group  $\text{FL}_b$  is quite well understood by Haefliger's work. For  $b = 1$ , we refer to [20] for more specific information.

We begin with the elementary

**LEMMA 5.1.** *Let  $k \in \text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5))$  be a commutator. Then  $k$  extends to an automorphism of  $\#_{i=1}^b(S^2 \times D^6)$ .*

*Proof.* For the proof, we depict  $\#_{i=1}^b(S^2 \times S^5)$  as follows: Let  $V_i$ ,  $i = 1, \dots, b$ , be  $b$  copies of  $S^2 \times D^6$ , and we join  $V_i$  and  $V_{i+1}$  by a tube  $T_i \cong [-1, 1] \times D^7$ ,  $i = 1, \dots, b-1$ . The result is a manifold  $W$  whose boundary is isomorphic to  $\#_{i=1}^b(S^2 \times S^5)$ . We make the following normalizations: Write  $\partial V_i$  as  $(S^2 \times D_+^i) \cup (S^2 \times D_-^i)$ , let  $n_i$  and  $s_i$  be the centers of  $D_+^i$  and  $D_-^i$ , respectively, and set  $S_+^i := S^2 \times n_i$  and  $S_-^i := S^2 \times s_i$ ,  $i = 1, \dots, b$ . Choose furthermore points  $e_i \neq w_i$  in  $(S^2 \times D_+^i) \cap (S^2 \times D_-^i)$ ,  $i = 1, \dots, b$ , and suppose that  $\{-1\} \times D^7 \subset T_i$  is attached to a disc around  $w_i$  in  $\partial V_i$  and  $\{1\} \times D^7 \subset T_i$  to a disc around  $e_{i+1}$  in  $\partial V_{i+1}$ ,  $i = 1, \dots, b-1$ . Set  $T := \bigsqcup_{i=1}^{b-1} T_i$ .

Now, let  $k = f \circ g \circ f^{-1} \circ g^{-1}$  with  $f, g \in \text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5))$ . As  $H_2(h, \mathbf{Z})$  is the identity for every element  $h \in \text{Aut}_0^{\text{PL}}(\#_{i=1}^b(S^2 \times S^5))$  and  $S_\pm^i$ ,  $i = 1, \dots, b$ , both represent the same basis for  $H_2(\partial W, \mathbf{Z})$ ,  $h$  is isotopic to a map  $h'$  which satisfies either assumption (A) or (B) below.