4. Prime parts of torsion numbers

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4. Prime parts of torsion numbers

We recall Jensen's formula, a short argument for which can be found in [Yo86].

LEMMA 4.1 [Jensen's formula]. *For any complex number* α,

$$\int_0^1 \log\left|\alpha - e^{2\pi i\theta}\right| d\theta = \log\max\{1, |\alpha|\}.$$

By Lemma 4.1 the Mahler measure $M(f)$ of a nonzero polynomial with complex coefficients can be computed as

$$\exp\int_0^1 \log\left|f(e^{2\pi i\theta})\right| d\theta.$$

This observation motivated the definition of Mahler measure for polynomials in several variables. (See [Bo81] or [EW99], for example.)

In [EF96], [Ev99] G.R. Everest and B. Ní Fhlathúin proved a p-adic analogue of Jensen's formula, which we describe. Assume that α is an algebraic integer lying in a finite extension K of $\mathbf{Q}$. For every prime p there is a p-adic absolute value $|\cdot|_p$, the usual Archimedean absolute value corresponding to ∞. We recall the definition (see [La65] for more details): If p is a prime number, then $|p^r m/n|_p = 1/p^r$, where r is an integer, and m, n are nonzero integers that are not divisible by p. By convention, $|0|_p = 0$. Each $|\cdot|_p$ extends to an absolute value $|\cdot|_v$ on K. Let Ω_v denote the smallest field which is algebraically closed and complete with respect to $|\cdot|_v$. Let $\mathbf{T}_v$ denote the closure of the group of all roots of unity, which is in general locally compact. Note that if $p = \infty$, then $\Omega_v = \mathbf{C}$ and $\mathbf{T}_v = \mathbf{T}$. Everest and Fhlathúin define

$$M_{\mathbf{T}_v}(t-\alpha) = \exp\int_{\mathbf{T}_v} \log|t-\alpha|_v d\mu = \exp\lim_{r\to\infty}\frac{1}{r}\sum_{\zeta^r=1}\log|\zeta-\alpha|_v.$$

Here $\int$ denotes the Shnirelman integral, given by the limit of sums at the right, where one skips over the undefined summands. The above integral exists even if $\alpha \in \mathbf{T}_v$, in which case it can be shown to be zero. Moreover, one has

$$\int_{\mathbf{T}_v} \log|t-\alpha|_v d\mu = \log\max\{1, |\alpha|_v\}, \tag{4.1}$$

which Everest and Fhlathúin refer to as a p-adic analogue of Jensen's formula.

Recall that $b_r^{(p)}$ denotes the *p-component* of b_r, the largest power of p that divides b_r. The *content* of $f \in \mathbf{Z}[t]$ is the greatest common divisor of the coefficients. Using (4.1) we will prove

THEOREM 4.2. *Let (G, χ) be an augmented group, and let p be a prime.*

(i) *If $\mathcal{M}$ has a square matrix presentation and $\Delta(t) \neq 0$, then the sequence $\{b_{r_k}\}$ of pure torsion numbers satisfies*

$$\lim_{r_k \to \infty} \left(b_{r_k}^{(p)}\right)^{1/r_k} = (\text{content } \Delta)^{(p)} .$$

(ii) *If $\mathcal{M}$ is a direct sum of cyclic modules, then the sequence of all torsion numbers satisfies*

$$\lim_{r \to \infty} \left(b_r^{(p)}\right)^{1/r} = (\text{content } \Delta)^{(p)} .$$

(iii) *If $\mathcal{M}$ is torsion free as an abelian group, then*

$$\lim_{r \to \infty} \left(b_r^{(p)}\right)^{1/r} = 1 .$$

EXAMPLE 4.3. For any positive integer m, consider the augmented group (G, χ) where G is the Baumslag-Solitar group $\langle x, y \mid y^m x = x y^m \rangle$ and $\chi \colon G \to \mathbf{Z}$ maps $x \mapsto 1$ and $y \mapsto 0$. One verifies that $\mathcal{M} \cong \mathcal{R}_1/(m(t-1))$. The quotient module $\mathcal{M}_r$ is isomorphic to $\mathbf{Z}^r/A_r\mathbf{Z}^r$, where

$$A_r = \begin{pmatrix} m & 0 & 0 & 0 & \cdots & -m \\ -m & m & 0 & \cdots & & 0 \\ 0 & -m & m & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & & \cdots & -m & m \end{pmatrix} .$$

The matrix is equivalent by elementary row and column operations to the diagonal matrix

$$\begin{pmatrix} m & & & \\ & \ddots & & \\ & & m & \\ & & & 0 \end{pmatrix} .$$

Hence $\mathcal{M}_r \cong \mathbf{Z} \oplus (\mathbf{Z}/m)^{r-1}$, and so $b_r = m^{r-1}$ for all r. Consequently,

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = m^{(p)} .$$

The Alexander polynomial of any knot is nonzero, and its coefficients are relatively prime. Hence the following corollary is immediate from Theorem 4.2 (iii).

COROLLARY 4.4. *For any knot k and prime p,*

$$\lim_{r\to\infty} (b_r^{(p)})^{1/r} = 1\,.$$

Theorem 2.10 and Corollary 4.4 imply that whenever the Alexander polynomial of k has Mahler measure greater than 1, infinitely many distinct primes occur in the factorization of the torsion numbers b_r. In other words, the homology groups $H_1(M_r, \mathbf{Z})$ display nontrivial p-torsion for infinitely many primes p. Since the sequence $\{b_r\}$ is a division sequence, the number of prime factors of b_r is unbounded.

What about the case in which the Alexander polynomial of k has Mahler measure equal to 1 ? The argument of Section 5.7 of [Go72] shows that the number of prime factors remains unbounded as long as the Alexander polynomial does not divide $t^M - 1$ for any M. If it does divide, then the torsion numbers b_r are periodic by Section 5.3 of [Go72] (see also Corollary 2.2 of [SiWi00]). Hence we obtain

COROLLARY 4.5. *For any knot, either the torsion numbers b_r are periodic or else for any $N > 0$ there exists an r such that the factorization of b_r has at least N distinct primes.*

The proof of Theorem 4.2 requires the following lemma.

LEMMA 4.6. *If $f(t) = c_0 t^n + \cdots + c_{n-1}t + c_n$ is a nonzero polynomial in $\mathbf{Z}[t]$ with roots $\lambda_1 \cdots, \lambda_n$ (not necessarily distinct) in Ω_v, then*

$$|c_0|_v \prod_{i=1}^{n} \max\{1, |\lambda_i|_v\} = |\text{content}\, f|_v\,.$$

Proof. The argument that we present is found in [LW88]. Set $a_j = c_j/c_0$ for $0 \le j \le n$, so $f(t) = c_0(t^n + a_1 t^{n-1} + \cdots + a_n)$. Each a_j is an elementary symmetric function of the roots λ_i, namely the sum of products of roots taken j at a time. Using the ultrametric property

$$|x+y|_v = \max\{|x|_v, |y|_v\}\,,$$

we see that if exactly k values of $|\lambda_i|_v$ are greater than 1, then

$$\max_j |a_j|_v = |a_k|_v = \prod_{j=1}^{n} \max\{1, |\lambda_j|_v\}\,.$$

But

$$\max_j |a_j|_v = \max\{1, \left|\frac{c_1}{c_0}\right|_v, \dots, \left|\frac{c_n}{c_0}\right|_v\} = \frac{|\text{content } f|_v}{|c_0|_v}.$$

Hence the lemma is proved. □

Proof of Theorem 4.2. In case (i), the pure torsion number b_{r_k} is equal to $\left|\prod_{\zeta^{r_k}=1} \Delta(\zeta)\right|$. We have

$$|b_{r_k}|_v = \left|\prod_{\zeta^{r_k}=1} \Delta(\zeta)\right|_v = |c_0|_v^{r_k} \prod_{\zeta^{r_k}=1} \prod_{j=1}^{n} |\zeta - \lambda_j|_v,$$

where c_0 is the leading coefficient of Δ and $\lambda_1, \dots \lambda_n$ are its roots. Hence

$$\begin{aligned}|b_{r_k}|_v^{1/r_k} &= |c_0|_v \prod_{\zeta^{r_k}=1} \prod_{j=1}^{n} |\zeta - \lambda_j|_v^{1/r_k} \\ &= |c_0|_v \prod_{j=1}^{n} \exp\Big(\frac{1}{r_k} \sum_{\zeta^{r_k}=1} \log|\zeta - \lambda_j|\Big),\end{aligned}$$

so that

$$\lim_{r_k \to \infty} |b_{r_k}|_v^{1/r_k} = |c_0|_v \prod_{j=1}^{n} \exp \int_{\mathbf{T}_v} \log|t - \lambda_j|_v \, d\mu,$$

which by equation (4.1) is equal to

$$|c_0|_v \prod_{j=1}^{n} \max\{1, |\lambda_j|_v\}.$$

By Lemma 4.6 this is equal to $|\text{content } \Delta|_v$. But for integers n we have $n^{(p)} = |n|_v^{-1}$.

Now suppose $\mathcal{M}$ is cyclic. As in the proof of Theorem 3.8, we let γ be the cyclotomic order of Δ and consider the subsequence of b_r with r in a fixed congruence class modulo γ. Then starting with Theorem 3.3 we may apply the argument above with Δ/Φ in place of Δ to show that the limit of $(|b_r|^{(p)})^{1/r}$ along this subsequence is the p-component of the content of Δ/Φ. But content is multiplicative and cyclotomic polynomials have content 1, so the limit along all congruence classes is (content $\Delta)^{(p)}$. The result is immediate for direct sums of cyclic modules.

Finally, we can extend the result when $\mathcal{M}$ is torsion-free as an abelian group using Theorem 3.6. But for this case the content of Δ is 1. □