2. A GENERAL STEINITZ ISOMORPHISM THEOREM

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2. A GENERAL STEINITZ ISOMORPHISM THEOREM

The classical Steinitz Isomorphism Theorem states that, for any nonzero ideals I, J in a Dedekind domain R, there is an R-isomorphism $I \oplus J \cong R \oplus IJ$. In view of the great importance of this theorem on the structure of f. g. modules over Dedekind rings, it is natural to try to extend the result to, say, invertible ideals over other classes of rings. In this section, we generalize the work of Heitmann and Levy [HL], and carry out the extension of the Steinitz Isomorphism Theorem to rings with the $1\frac{1}{2}$ generator property having small 0-divisors.

We begin with the following lemma that offers several characterizations for rings having small 0-divisors. For such rings, this result shows in particular that the regularity assumption on the element a in the definition of the $1\frac{1}{2}$ generator property can actually be dropped.

LEMMA 1. *For any ring R, the following are equivalent:*

(1) *R has small 0-divisors.*

(2) $\mathcal{Z}(R) \subseteq \operatorname{rad}(R)$.

(3) $1 + \mathcal{Z}(R) \subseteq U(R)$ *(the group of units of R).*

(4) *For any invertible ideal $I \subseteq R$, any element $a \in I \setminus (\operatorname{rad} R)I$ is regular.*

Proof. (1) $\Longleftrightarrow$ (2) is clear from the fact that $\operatorname{rad}(R)$ is the largest small ideal in R. Next, (2) $\Longleftrightarrow$ (3) follows from the fact that $\mathcal{Z}(R)$ is closed under multiplication by elements of R, and the fact that an element $a \in R$ lies in $\operatorname{rad}(R)$ if and only if $1 + Ra \subseteq U(R)$.

(4) $\Longrightarrow$ (2). If $\mathcal{Z}(R) \nsubseteq \operatorname{rad}(R)$, fix an element $a \in \mathcal{Z}(R) \setminus \operatorname{rad}(R)$. Then for the invertible ideal $I = R$, the element $a \in I \setminus (\operatorname{rad} R)I$ fails to be regular.

(2) $\Longrightarrow$ (4). Assume that $\mathcal{Z}(R) \subseteq \operatorname{rad}(R)$, and consider an element a as in (4). Let $Q(R)$ be the total ring of quotients of R, and let

$$I^{-1} = \{q \in Q(R) : qI \subseteq R\}.$$

Since $II^{-1} = R$, $a \in I \setminus (\operatorname{rad} R)I$ implies that $aI^{-1} \nsubseteq \operatorname{rad}(R)$, and of course $aI^{-1} \subseteq R$. If a is not regular, then $ra = 0$ for some nonzero $r \in R$. But then $r(aI^{-1}) = 0$ and so $aI^{-1} \subseteq \mathcal{Z}(R) \subseteq \operatorname{rad}(R)$, which is not the case. Thus, a must be regular. $\square$

The characterizations in Lemma 1 enable us to give some quick examples of rings with small 0-divisors besides obvious ones such as integral domains and local rings. For instance, any ring R whose 0-divisors are nilpotent is a ring with small 0-divisors. This is the class of rings expressible as A/Q where Q is a primary ideal in the ring A.

Rings having small 0-divisors, as characterized in (2) above, were first used systematically by Kaplansky in [Ka_1]; see his theorems (3.2), (5.1), (12.1), etc., which all depended on the property (2). It is perhaps tempting to think that there is a some kind of "duality" between the property (1) "0-divisors in R are small" and the property "non 0-divisors in R are large" (where "large" is in the sense of "essential"). However, the latter is true in *any* commutative ring, while the former is only a condition.

LEMMA 2. *Let R be a ring with small 0-divisors, and I be an invertible ideal in R. Then:*

(1) *$I = Ra_1 + \cdots + Ra_n$, where each $a_i \in I \setminus (\text{rad } R)I$ (and hence regular by Lemma 1).*

(2) *If, moreover, R has the $1\frac{1}{2}$ generator property, then $I = Ra + Rc$, where $a, c \in I \setminus (\text{rad } R)I$ (and hence regular by Lemma 1), and a can be arbitrarily prescribed.*

Proof. (1) It is well known that I is finitely generated, say $I = Ra_1 + \cdots + Ra_m$. After relabelling the indices, we may assume that $a_1, \ldots, a_n \notin (\text{rad } R)I$, while $a_{n+1}, \ldots, a_m \in (\text{rad } R)I$. Then we have

$$I = Ra_1 + \cdots + Ra_n + (\text{rad } R)I,$$

and so Nakayama's Lemma implies that $I = Ra_1 + \cdots + Ra_n$, as desired[2]).

(2) Start with any element $a \in I \setminus (\text{rad } R)I$. Since a is automatically regular by the condition (4) in Lemma 1, the definition of the $1\frac{1}{2}$ generator property guarantees that $I = Ra + Rb$ for some $b \in I$. If $b \notin (\text{rad } R)I$, we are done by taking $c = b$. Otherwise, Nakayama's Lemma implies that $I = Ra$, and we can just take $c = a$. □

Using Lemma 2, we obtain the following result in generalization of [HL: (4.1)]. Note that, in contrast to [HL], R is *not* assumed to be a Prüfer ring in this result.

[2]) In connection to the conclusion of this part, it is relevant to point out that, in general, an invertible ideal in a Prüfer ring need not be generated by regular elements: for such an example, see [AP].

GENERAL STEINITZ ISOMORPHISM THEOREM. *Let R be a ring with the $1\frac{1}{2}$ generator property having small 0-divisors. For any two invertible ideals I, J in R, we have an R-module isomorphism $I \oplus J \cong R \oplus IJ$.*

Proof. In general, any invertible ideal is regular (see [La$_2$: (2.17)]), so we can fix a regular element $x \in I$. Let $I_1 := xI^{-1} \subseteq II^{-1} = R$. Then I_1 is an invertible ideal (with inverse $x^{-1}I$), and hence so is I_1J.

CASE 1. $J \not\subseteq \operatorname{rad}(R)$. Since I_1 is invertible, this implies $I_1J \not\subseteq (\operatorname{rad} R)I_1$. Fix an element $a \in I_1J \setminus (\operatorname{rad} R)I_1$. Applying Lemma 2(2) to the invertible ideal I_1, we can write $I_1 = Ra + Rc$ for a suitable *regular* element $c \in I_1$. In particular, we have $I_1 = I_1J + Rc$. Multiplying this equation (in $Q(R)$) by $I_1^{-1} = x^{-1}I$, we get $R = J + x^{-1}cI$. Now $x^{-1}cI \cong I$ (as R-modules) since x and c are both regular. Thus,

$$I \oplus J \cong x^{-1}cI \oplus J \cong R \oplus x^{-1}cIJ \cong R \oplus IJ.$$

Here, the second isomorphism holds since the ideal $K := x^{-1}cI$ is comaximal with J. (The isomorphism is simply given by splitting the obvious epimorphism from $K \oplus J$ to $K + J = R$, the kernel of which is isomorphic to $K \cap J = K \cdot J$.)

CASE 2. $J \subseteq \operatorname{rad}(R)$. Since J is invertible, there exist elements $q_j \in J^{-1}$ such that $\sum_j Jq_j = R$. For a suitable index i, we have $Jq_i \not\subseteq \operatorname{rad}(R)$. Write $q_i = s_i/s$, where $s_i \in R$, and s is a regular element in R. Arguing as in the proof of (2) $\Longrightarrow$ (4) in Lemma 1, we see that s_i must be regular, so q_i is a unit in $Q(R)$. By replacing J with the isomorphic R-ideal $Jq_i \not\subseteq \operatorname{rad}(R)$, we are back to Case 1. $\square$

It is worthwhile to point out that the Steinitz Isomorphism Theorem does not hold in general for invertible ideals even in a Prüfer domain. For, if $I \oplus J \cong R \oplus IJ$ holds for all invertible ideals $I, J \subseteq R$, then, taking J to be isomorphic to I^{-1}, we get readily that $I \oplus I^{-1} \cong R \oplus R$, which would imply that I is 2-generated. But according to Schülting [Sch], there exists a (2-dimensional) Prüfer domain R with an (invertible) ideal that is 3-generated but not 2-generated. Thus, the Steinitz Isomorphism cannot hold for R.

3. DECOMPOSITION OF FINITELY GENERATED PROJECTIVE MODULES

In this section, we study the structure of f.g. projective modules over Prüfer rings with small 0-divisors. The following result may be viewed as