3. DECOMPOSITION OF FINITELY GENERATED PROJECTIVE MODULES

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General Steinitz Isomorphism Theorem. *Let R be a ring with the $1\frac{1}{2}$ generator property having small 0-divisors. For any two invertible ideals I, J in R, we have an R-module isomorphism $I \oplus J \cong R \oplus IJ$.*

Proof. In general, any invertible ideal is regular (see [La$_2$: (2.17)]), so we can fix a regular element $x \in I$. Let $I_1 := xI^{-1} \subseteq II^{-1} = R$. Then I_1 is an invertible ideal (with inverse $x^{-1}I$), and hence so is $I_1 J$.

Case 1. $J \not\subseteq \operatorname{rad}(R)$. Since I_1 is invertible, this implies $I_1 J \not\subseteq (\operatorname{rad} R) I_1$. Fix an element $a \in I_1 J \setminus (\operatorname{rad} R) I_1$. Applying Lemma 2(2) to the invertible ideal I_1, we can write $I_1 = Ra + Rc$ for a suitable *regular* element $c \in I_1$. In particular, we have $I_1 = I_1 J + Rc$. Multiplying this equation (in $Q(R)$) by $I_1^{-1} = x^{-1}I$, we get $R = J + x^{-1}cI$. Now $x^{-1}cI \cong I$ (as R-modules) since x and c are both regular. Thus,

$$I \oplus J \cong x^{-1}cI \oplus J \cong R \oplus x^{-1}cIJ \cong R \oplus IJ.$$

Here, the second isomorphism holds since the ideal $K := x^{-1}cI$ is comaximal with J. (The isomorphism is simply given by splitting the obvious epimorphism from $K \oplus J$ to $K + J = R$, the kernel of which is isomorphic to $K \cap J = K \cdot J$.)

Case 2. $J \subseteq \operatorname{rad}(R)$. Since J is invertible, there exist elements $q_j \in J^{-1}$ such that $\sum_j J q_j = R$. For a suitable index i, we have $J q_i \not\subseteq \operatorname{rad}(R)$. Write $q_i = s_i / s$, where $s_i \in R$, and s is a regular element in R. Arguing as in the proof of (2) $\Longrightarrow$ (4) in Lemma 1, we see that s_i must be regular, so q_i is a unit in $Q(R)$. By replacing J with the isomorphic R-ideal $J q_i \not\subseteq \operatorname{rad}(R)$, we are back to Case 1. $\square$

It is worthwhile to point out that the Steinitz Isomorphism Theorem does not hold in general for invertible ideals even in a Prüfer domain. For, if $I \oplus J \cong R \oplus IJ$ holds for all invertible ideals $I, J \subseteq R$, then, taking J to be isomorphic to I^{-1}, we get readily that $I \oplus I^{-1} \cong R \oplus R$, which would imply that I is 2-generated. But according to Schülting [Sch], there exists a (2-dimensional) Prüfer domain R with an (invertible) ideal that is 3-generated but not 2-generated. Thus, the Steinitz Isomorphism cannot hold for R.

3. Decomposition of finitely generated projective modules

In this section, we study the structure of f.g. projective modules over Prüfer rings with small 0-divisors. The following result may be viewed as an analogue of the theorem of Cartan and Eilenberg [CE: Ch. I, Prop. 6.1] (see also [La$_2$: (2.29)]) on the decomposition of f. g. projective modules over semihereditary rings.

THEOREM 3. *Let R be a Prüfer rings with small 0-divisors. Then any f. g. projective R-module P is isomorphic to a finite direct sum of invertible ideals in R.*

Proof. Note first that the second assumption on R implies that R is *connected.* In fact, if $e \in R$ is an idempotent $\neq 1$, then for $f = 1 - e \neq 0$, we have $fe = 0$, so $e \in \mathcal{Z}(R)$. Now $R = Re + Rf \Longrightarrow R = Rf$. Therefore, f is a unit, and so $f = 1$ and $e = 0$. The connectedness of R implies that *any f. g. projective R-module has a constant rank.*

We may assume in the following that $P \neq 0$. By a theorem of Bass [La$_1$: (24.7)], the radical $\mathrm{rad}(P)$ of P is a proper submodule, so there exists an element $a \in P \setminus \mathrm{rad}(P)$. Now choose a module Q such that $F := P \oplus Q$ is a free R-module. Since $\mathrm{rad}(F) = \mathrm{rad}(P) \oplus \mathrm{rad}(Q)$, we have

$$a \notin \mathrm{rad}(F) = \mathrm{rad}(R) \cdot F$$

by [La$_1$: (4.6) (3)]. Thus, if $a = \sum_i a_i e_i$ with respect to a basis $\{e_i\}$ of F, there exists an index j for which $a_j \notin \mathrm{rad}(R)$. Since $\mathcal{Z}(R) \subseteq \mathrm{rad}(R)$, it follows that a_j is a *regular* element in R.

Let $\pi_j \colon F \to R$ denote the j^{th} coordinate projection on F. Since P is f. g., $\pi_j(P)$ is a f. g. ideal containing a regular element a_j. Thus, $\pi_j(P)$ is an invertible ideal, and hence a projective R-module. This implies that the surjection $\pi_j \colon P \to \pi_j(P)$ splits, so P has a direct summand isomorphic to the invertible ideal $\pi_j(P)$. By invoking an inductive hypothesis on the rank of P, it follows that P is isomorphic to a finite direct sum of invertible ideals of R. $\square$

REMARK. In the argument above, we used the f. g. assumption on P only in the third paragraph. If P is *not* f. g., the work in the second paragraph above can still be used as a beginning step for arguing that P contains a direct summand isomorphic to an invertible ideal. For this more sophisticated argument, see the proof of Theorem 9 below.

Note that the above theorem implies, in particular, that $\mathrm{Pic}(R)$ (the Picard group of R) is the same as $\mathrm{Cl}(R)$ (the ideal class group of R). This shows, incidentally, that the theorem may not hold if we do not assume that R has small 0-divisors. In fact, there exist examples of classical rings of quotients R whose Picard group is nontrivial (see, e.g. [La$_2$: (2.22)]); such rings are of course Prüfer rings with trivial ideal class groups.

We are now in a position to prove Theorem A stated in the Introduction. In the proof below, *R is assumed to be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors.*

Proof of Theorem A (sketch). For any f. g. projective R-module J of rank 1, let $\{J\}$ denote the isomorphism class of J, viewed as an element in the Picard group $\mathrm{Pic}(R)$. For any f. g. projective R-module P, let $[P] \in K_0(R)$, and let $\overline{[P]}$ denote its image in $\widetilde{K}_0(R) = K_0(R)/\mathbf{Z} \cdot [R]$. We define a map $\alpha\colon \mathrm{Pic}(R) \to \widetilde{K}_0(R)$ by $\alpha\{J\} = \overline{[J]}$. Note that, for any ideals $I, J \subseteq R$ with J invertible, the R-module J is (projective and hence) flat, so we have an R-module isomorphism $I \otimes_R J \cong IJ$. This and the General Steinitz Isomorphism Theorem readily imply that α is a group homomorphism. Using Theorem 3, we see that α is surjective. Finally, the usual exterior algebra argument gives the injectivity of α. Thus, $\alpha\colon \mathrm{Pic}(R) \to \widetilde{K}_0(R)$ is a group isomorphism, as desired. □

Having completed the proof of Theorem A, we can easily derive the following corollary on the structure and classification of f. g. projective modules over the Prüfer rings in question. Its proof is essentially the same as that in the classical case of Dedekind domains, so we shall omit it.

COROLLARY 4. *Let R be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors. Then any f. g. projective R-module P of rank n is isomorphic to $R^{n-1} \oplus I$ where I is an invertible ideal, and the isomorphism class of P is determined by that of I and the rank n. In particular, f. g. projective R-modules satisfy the cancellation law, and f. g. stably free R-modules are free.*

4. RINGS WITH THE STRONG $1\frac{1}{2}$-GENERATOR PROPERTY

In view of the results in the previous sections, it is of interest to find examples of rings satisfying the $1\frac{1}{2}$ generator property. As it turns out, it is actually easier to name some rings that satisfy a stronger property: let us say that a ring R has the *strong $1\frac{1}{2}$ generator property* if any invertible ideal $I \subseteq R$ is generated by two elements, the first of which can be any prescribed