2. Nonamenability for graphs

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examples of such subgroups are also possible. For instance, if F is a free group of finite rank and $\phi\colon F \to F$ is an atoroidal automorphism, then the mapping torus group of ϕ

$$M_\phi = \langle F, t \mid t^{-1}ft = \phi(f) \text{ for all } f \in F \rangle$$

is word-hyperbolic [8, 13]. In this case $M_\phi/F \simeq \mathbf{Z}$ and hence the Schreier graph for M_ϕ relative to F is amenable.

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2. NONAMENABILITY FOR GRAPHS

Let X be a connected graph of bounded degree. We define the *spectral radius* $\rho(X)$ of X as

$$\rho(X) := \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}$$

where x, y are two vertices of X and $p^{(n)}(x,y)$ is the probability that an n-step simple random walk starting at x will end up at y. It is well-known that $\rho(X) \leq 1$ and that the definition of $\rho(X)$ does not depend on the choice of x, y.

DEFINITION 2.1 (Amenability for graphs). A connected graph X of bounded degree is said to be *amenable* if $\rho(X) = 1$ and *nonamenable* if $\rho(X) < 1$.

It is also well-known that nonamenability of X implies that X is *transient*, that is that for a simple random walk on X the probability of ever returning to the basepoint is less than 1 (see for example Theorem 51 of [16]). We refer the reader to [16, 71, 72] for comprehensive background information about random walks on graphs and for further references on this topic.

CONVENTION 2.2. Let X be a connected graph of bounded degree with the simplicial metric d. For a finite nonempty subset $S \subset VX$ we will denote by $|S|$ the number of elements in S.

If S is a finite subset of the vertex set of X and $k \geq 1$ is an integer, we will denote by $\mathcal{N}_k^X(S) = \mathcal{N}_k(S)$ the set of all vertices v of X such that $d(v, S) \leq k$. Also, we will denote $\eth^X S = \eth S := \mathcal{N}_1(S) - S$.

The number

$$\iota(X) := \inf\{\frac{|\eth S|}{|S|} \mid S \text{ is a finite nonempty subset of the vertex set of } X\}$$

is called the *Cheeger constant* or the *isoperimetric constant* of X.

There are many alternative definitions of nonamenability:

PROPOSITION 2.3. *Let X be a connected graph of bounded degree with the simplicial metric d. Then the following conditions are equivalent:*

1. *The graph X is nonamenable.*
2. (Følner Criterion) *We have $\iota(X) > 0$.*
3. (Gromov's Doubling Condition) *There is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq 2|S|.$$
4. *For any integer $q > 1$ there is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq q|S|.$$
5. *For some $0 < \sigma < 1$ we have $p^{(n)}(x, y) = o(\sigma^n)$ for any $x, y \in VX$.*
6. *Let $W(X)$ be the pseudogroup of "bounded perturbations of the identity", that is $W(X)$ consists of all bijections ϕ between subsets of VX such that*
$$\sup_{x \in dom(\phi)} d(x, \phi(x)) < \infty.$$
Then $W(X)$ admits a "paradoxical decomposition", that is there exist nonempty subsets Y_1, Y_2 of VX and $\phi_1 \colon Y_1 \to VX$, $\phi_2 \colon Y_2 \to VX$ such that $\phi_1, \phi_2 \in W(X)$, $VX = Y_1 \sqcup Y_2$ and $\phi_1(Y_1) = \phi_1(Y_2) = VX$.
7. ("Grasshopper Criterion") *There exists a map $\phi \colon VX \to VX$ such that*
$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$
and such that for any $x \in VX$ we have $\left|\phi^{-1}(x)\right| \geq 2$.
8. *There exists a map $\phi \colon VX \to VX$ such that*
$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$
and such that for any $x \in VX$ we have $\left|\phi^{-1}(x)\right| = 2$.

9. *The bottom of the spectrum for the combinatorial Laplacian operator on* X *is* > 0 *(see* [21] *for the precise definitions).*
10. *We have* $H_0^{uf}(X) = 0$ *(see* [9] *for the precise definition of the* uniformly finite *homology groups* H_i^{uf} *).*
11. *We have* $H_0^{(l_p)}(X) = 0$ *for any* $1 < p < \infty$ *(see* [24] *for the precise definition of* $H_i^{(l_p)}$ *).*

All of the above statements are well-known, but we will still provide some sample references. The fact that (1), (2), (5) and (6) are equivalent is stated in Theorem 51 of [16]. The fact that (3), (4), (6), (7) and (8) are equivalent follows from Theorem 32 of [16]. The equivalence of (2) and (9) is due to J. Dodziuk [21]. J. Block and S. Weinberger [9] established the equivalence of (2) and (10). Finally, G. Elek [24] proved that (2) is equivalent to (11).

One can characterize amenability of regular graphs in terms of cogrowth.

DEFINITION 2.4. Let X be a connected graph of bounded degree with a base-vertex x_0. Let $a_n = a_n(X, x_0)$ be the number of reduced edge-paths of length n from x_0 to x_0. Let $b_n = b_n(X, x_0)$ be the number of all edge-paths of length n from x_0 to x_0. Set

$$\alpha(X) := \limsup_{n\to\infty} \sqrt[n]{a_n} \qquad \text{and} \qquad \beta(X) := \limsup_{n\to\infty} \sqrt[n]{b_n}\,.$$

Then we will call $\alpha(X)$ the *cogrowth rate* of X and we will call $\beta(X)$ the *non-reduced cogrowth rate* of X. These definitions are independent of the choice of x_0.

It is easy to see that for a d-regular connected graph X we have $\alpha(X) \le d-1$ and $\beta(X) \le d$. Moreover, $\rho(X) = \frac{\beta(X)}{d}$. The following result was originally proved by R. Grigorchuk [39] and J. Cohen [19] for the Cayley graphs of finitely generated groups and by L. Bartholdi [5] for arbitrary regular graphs.

THEOREM 2.5 ([5]). *Let* X *be a connected* d*-regular graph with* $d \ge 3$*. Set* $\alpha = \alpha(X)$*,* $\beta = \beta(X)$ *and* $\rho = \rho(X)$*. Then*

$$\rho = \begin{cases} \frac{2\sqrt{d-1}}{d} & \text{if } 1 \le \alpha \le \sqrt{d-1} \\ \frac{\sqrt{d-1}}{d}\left(\frac{\sqrt{d-1}}{\alpha} + \frac{\alpha}{\sqrt{d-1}}\right) & \text{if } \sqrt{d-1} \le \alpha \le d-1\,. \end{cases}$$

In particular $\rho < 1 \iff \alpha < d-1 \iff \beta < d$.