4. Quasiconvex subgroups of hyperbolic groups

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4. QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

Detailed background information on quasiconvex subgroups of hyperbolic groups can be found in [1, 4, 20, 31, 38, 34, 32, 51, 54, 68] and other sources.

CONVENTION 4.1. Suppose G is a finitely generated group with a fixed finite generating set A. Let $X = \Gamma(G, A)$ be the Cayley graph of G with respect to A. We will denote the word-metric corresponding to A on X by d_A. Also, for $g \in G$ we will denote $|g|_A := d_A(1, g)$. For a word w in the alphabet $A \cup A^{-1}$ we will denote by $\overline{w}$ the element of G represented by w.

DEFINITION 4.2 (Quasiconvexity). For $\epsilon \geq 0$ a subset Z of a metric space (X, d) is *ϵ-quasiconvex* if, for any $z_1, z_2 \in Z$ and any geodesic $[z_1, z_2]$ in X, the segment $[z_1, z_2]$ is contained in the closed ϵ-neighborhood of Z. A subset $Z \subseteq X$ is *quasiconvex* if it is ϵ-quasiconvex for some $\epsilon \geq 0$.

If G is a finitely generated group and A is a finite generating set of G, a subgroup $H \leq G$ is *quasiconvex in G with respect to A* if $H \subseteq \Gamma(G, A)$ is a quasiconvex subset.

It turns out [20, 32, 4, 31] that for subgroups of word-hyperbolic groups quasiconvexity is independent of the choice of a finite generating set for the ambient group. Thus a subgroup H of a hyperbolic group G is termed *quasiconvex* if $H \subseteq \Gamma(G, A)$ is quasiconvex for some finite generating set A of G.

We summarize some well-known basic facts regarding quasiconvex subgroups and provide some sample references:

PROPOSITION 4.3. *Let G be a word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G, A)$ be the Cayley graph of G with the word-metric d_A induced by A. Then:*

1. *If $H \leq G$ is a subgroup, then either H is virtually cyclic (in which case H is called* elementary*) or H contains a free subgroup F of rank two which is quasiconvex in G (in this case H is said to be* nonelementary*)* [20, 32].
2. *Every cyclic subgroup of G is quasiconvex in G* [1, 20, 32].
3. *If $H \leq G$ is quasiconvex then H is finitely presentable and word-hyperbolic* [1, 20, 32].

4. *Suppose $H \leq G$ is generated by a finite set Q inducing the word-metric d_Q on H. Then H is quasiconvex in G if and only if there is a $C > 0$ such that for any $h_1, h_2 \in H$*

$$d_Q(h_1, h_2) \leq C d_A(h_1, h_2)$$

(*see* [20, 32, 4, 31]).
5. *The set $\mathcal{L}$ of all A-geodesic words is a regular language that provides a bi-automatic structure for G. Moreover, a subgroup $H \leq G$ is quasiconvex if and only if H is $\mathcal{L}$-rational, that is the set $\mathcal{L}_H = \{w \in \mathcal{L} \mid \overline{w} \in H\}$ is a regular language* [31].
6. *If $H_1, H_2 \leq G$ are quasiconvex, then $H_1 \cap H_2 \leq G$ is quasiconvex* [68].
7. [51, 46] *Let $C \leq B \leq G$ where B is quasiconvex in G (and hence B is hyperbolic) and C is quasiconvex in B. Then C is quasiconvex in G* [51, 46].
8. *Let $C \leq B \leq G$ where C is quasiconvex in G and where B is word-hyperbolic. Then C is quasiconvex in B* [51, 46].
9. *Suppose $H \leq G$ is an infinite quasiconvex subgroup. Then H has finite index in its* commensurator *$Comm_G(H)$ (see* [51]), *where $Comm_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty$ and $[g^{-1}Hg : g^{-1}Hg \cap H] < \infty\}$.*

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

5. Proof of the main result

Let G be a nonelementary word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G, A)$ be the Cayley graph of G with the word metric d_A. Let $\delta \geq 1$ be an integer such that the space $(\Gamma(G, A), d_A)$ is δ-hyperbolic. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G. These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

Lemma 5.1. *There exists an integer constant $K = K(G, H, A) > 0$ with the following properties.*

Assume $g \in G$ is shortest with respect to d_A in the coset class Hg. Then for any $h \in H$ we have $(g, h)_1 \leq K$ (and hence $(g, H)_1 \leq K$).