5. PROOF OF THE MAIN RESULT

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4. *Suppose* $H \leq G$ *is generated by a finite set* Q *inducing the word-metric* d_Q *on* H. *Then* H *is quasiconvex in* G *if and only if there is a* $C > 0$ *such that for any* $h_1, h_2 \in H$

$$d_Q(h_1, h_2) \leq C d_A(h_1, h_2)$$

(*see* [20, 32, 4, 31]).
5. *The set* $\mathcal{L}$ *of all* A*-geodesic words is a regular language that provides a bi-automatic structure for* G. *Moreover, a subgroup* $H \leq G$ *is quasiconvex if and only if* H *is* $\mathcal{L}$*-rational, that is the set* $\mathcal{L}_H = \{w \in \mathcal{L} \mid \overline{w} \in H\}$ *is a regular language* [31].
6. *If* $H_1, H_2 \leq G$ *are quasiconvex, then* $H_1 \cap H_2 \leq G$ *is quasiconvex* [68].
7. [51, 46] *Let* $C \leq B \leq G$ *where* B *is quasiconvex in* G *(and hence* B *is hyperbolic) and* C *is quasiconvex in* B. *Then* C *is quasiconvex in* G [51, 46].
8. *Let* $C \leq B \leq G$ *where* C *is quasiconvex in* G *and where* B *is word-hyperbolic. Then* C *is quasiconvex in* B [51, 46].
9. *Suppose* $H \leq G$ *is an infinite quasiconvex subgroup. Then* H *has finite index in its* commensurator $Comm_G(H)$ (*see* [51]), *where* $Comm_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty$ *and* $[g^{-1}Hg : g^{-1}Hg \cap H] < \infty\}$.

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

5. Proof of the main result

Let G be a nonelementary word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G, A)$ be the Cayley graph of G with the word metric d_A. Let $\delta \geq 1$ be an integer such that the space $(\Gamma(G, A), d_A)$ is δ-hyperbolic. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G. These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

LEMMA 5.1. *There exists an integer constant* $K = K(G, H, A) > 0$ *with the following properties.*

Assume $g \in G$ *is shortest with respect to* d_A *in the coset class* Hg. *Then for any* $h \in H$ *we have* $(g, h)_1 \leq K$ *(and hence* $(g, H)_1 \leq K$*).*

Proof. The conclusion of Lemma 5.1 follows directly from the proofs of Lemma 4.1 and Lemma 4.5 of [4]. We will present the argument for completeness. For the hyperbolic space $X = \Gamma(G,A)$ choose $\delta' \geq 0$ as in part 2 of Proposition 3.3. Let $\epsilon \geq 0$ be such that H is an ϵ-quasiconvex subset of X.

Let $g \in G$ be a shortest element of Hg, so that for any $h \in H$ we have $|hg|_A \leq |g|_A$. We claim that $(h,g)_1 \leq \epsilon + \delta'$ for any $h \in H$.

Suppose not, that is $(h,g)_1 > \epsilon + \delta'$ for some $h \in H$. Consider two geodesic segments $[1,g]$ and $[1,h]$ in X and let $t \in [1,h]$, $s \in [1,g]$ be such that $d_A(1,s) = d_A(1,t) = (h,g)_1$. Thus $d_A(s,t) \leq \delta'$ by the choice of δ'. Since H is ϵ-quasiconvex in X, there is $h' \in H$ such that $d_A(t,h') \leq \epsilon$. Then

$$\begin{aligned} \left|(h')^{-1}g\right|_A &= d_A(h',g) \leq d_A(h',t) + d_A(t,s) + d_A(s,g) \\ &\leq \epsilon + \delta + |g|_A - (h,g)_1 < |g|_A , \end{aligned}$$

which contradicts the assumption that g is shortest in Hg.

LEMMA 5.2. *Let $T_1, T_2 > 0$ be some positive numbers. Let $g \in G$ be such that $(g,H)_1 \leq T_1$ and $|g|_A > T_1 + T_2 + \delta$. Let $f \in G$ be such that $|f|_A \leq T_2$. Then $(gf,H)_1 \leq T_1 + \delta$.*

Proof. Note that $|g|_A = (g,gf)_1 + (1,gf)_g$. Since $(1,gf)_g \leq d(g,gf) = |f|_A \leq T_2$, we conclude that

$$(g,gf)_1 = |g|_A - (1,gf)_g > T_1 + T_2 + \delta - T_2 = T_1 + \delta .$$

Therefore for any $h \in H$ we have

$$T_1 + \delta \geq (g,h)_1 + \delta \geq \min\{(g,gf)_1, (gf,h)_1\}$$

and hence $(gf,h)_1 \leq T_1 + \delta$ because $(g,gf)_1 > T_1 + \delta$. Since $h \in H$ was arbitrary, this means that $(gf,H)_1 \leq T_1 + \delta$.

LEMMA 5.3. *Suppose $g_1, g_2 \in G$ are such that $Hg_1 = Hg_2$. Then there is $h \in H$ such that $hg_1 = g_2$ and that*

$$|h|_A \leq (g_1,H)_1 + (g_2,H)_1 .$$

Proof. Since $Hg_1 = Hg_2$, there is $h \in H$ with $hg_1 = g_2$. Hence

$$|h|_A = (h,g_2)_1 + (1,hg_1)_h = (h,g_2)_1 + (h^{-1},g_1)_1 \leq (g_2,H)_1 + (g_1,H)_1 .$$

Proof of Theorem 1.2. Let $K = K(G,H,A) > 0$ be the constant provided by Lemma 5.1. Put $Y = \Gamma(G,H,A)$. Thus Y is a connected $2m$-regular infinite graph, where m is the number of elements in A. Denote the simplicial metric on Y by d_Y.

Let N be the number of all elements $g \in G$ with $|g|_A \le 2K + 2\delta$. In particular Y has at most N vertices within distance $2K + 2\delta$ of the coset $H1 \in VY$.

Since G is nonelementary word-hyperbolic and thus nonamenable, the Cayley graph $X = \Gamma(G,A)$ is nonamenable. By part 4 of Proposition 2.3 there is a constant $k' > 0$ such that for any finite nonempty subset S of G the k'-neighborhood of S in X has at least $4N|S|$ vertices. Let N_1 be the number of elements of G of length at most $K+\delta+k'$. Choose $k'' > 1$ such that for any vertex $Hg \in VY$ with $d_Y(H1, Hg) \le K + \delta + k'$ the k''-neighborhood of Hg has at least $4N_1$ vertices. Such k'' exists since by assumption $[G : H] = \infty$ and hence the graph Y is infinite. Set $k := \max\{k', k''\}$.

Suppose now that $F \subset VY$ is a finite nonempy subset. Write $F = F_1 \sqcup F_2$ where F_1 is the intersection of F with the closed ball of radius $K + \delta + k'$ in Y.

If $|F_1| \ge |F|/2$, then $|F| \le 2N_1$ and the k-neighborhood of F in Y has at least $4N_1 \ge 2|F|$ vertices. Suppose now that $|F_1| < |F|/2$, so that $|F_2| \ge |F|/2$. Then

$$F_2 = \{Hg_1, \dots, Hg_t\}$$

where $|F_2| = t$ and where each $g_i \in G$ is shortest in Hg_i with $|g_i|_A > K+\delta+k'$. By Lemma 5.1 $(g_i, H)_1 \le K$. By Lemma 5.2 for any $f \in G$ with $|f|_A \le k'$ and for each $i = 1, \dots, t$ we have $(g_i f, H)_1 \le K + \delta$.

Let $S := \{g_1, \dots, g_t\}$ and let S' be the set of all vertices of X contained in the k'-neighborhood of S in X. By the choice of k' we have $|S'| \ge 4N|S| = 4N|F_2|$. On the other hand, Lemma 5.3 implies that if $g, g' \in S'$ are such that $Hg = Hg'$ then $hg = g'$ for some $h \in H$ with $|h|_A \le 2K + 2\delta$. By the choice of N this means that the set $F' := \{Hg \mid g \in S'\}$ contains at least

$$|S'|/N = 4N|F_2|/N = 4|F_2| \ge 2|F|$$

distinct elements. However, F' is obviously contained in the k-neighborhood of F in Y.

We have verified that for any finite nonempy subset $F \subseteq VY$ the k-neighborhood of F in Y contains at least $2|F|$ vertices. By the Doubling Condition (part 3 of Proposition 2.3) this implies that Y is nonamenable.

We can now obtain Corollary 1.4 stated in the Introduction.

COROLLARY 5.4. *Let $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$ be a nonelementary word-hyperbolic group and let $H \leq G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H. Let b_n be the number of all words in A of length n that represent elements of H. Then*

$$\limsup_{n \to \infty} \sqrt[n]{a_n} < 2k - 1$$

and

$$\limsup_{n \to \infty} \sqrt[n]{b_n} < 2k\,.$$

Proof. Note that $k \geq 2$ since G is nonelementary. Put $A = \{x_1, \dots, x_k\}$ and $Y = \Gamma(G, H, A)$. We choose $x_0 := H1 \in VY$ as the base-vertex of Y. Note that Y is $2k$-regular by construction. Also, for any vertex x of Y and any word w in $A \cup A^{-1}$ there is a unique path in Y with label w and origin x. The definition of Schreier coset graphs also implies that a word w represents an element of H if and only if the unique path in Y with origin x_0 and label w terminates at x_0. Therefore $a_n(Y)$ equals the number of freely reduced words in the alphabet $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H. Similarly, $b_n(Y)$ equals the number of all words in A of length n representing elements of H. By Theorem 1.2, Y is nonamenable. Hence by Theorem 2.5, $\alpha(Y) < 2k - 1$ and $\beta(Y) < 2k$, as required.

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