

4. The triple ratio on S

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

4. THE TRIPLE RATIO ON S

We return to the notation introduced in Sections 1 and 2.

For $z_1, z_2, z_3 \in \text{Mat}(p \times q, \mathbf{C})$ define, whenever it makes sense, the element $T(z_1, z_2, z_3) \in \text{GL}(q, \mathbf{C})$ by the following formula

$$(24) \quad \begin{aligned} T(z_1, z_2, z_3) &= k(z_1, z_2) k(z_3, z_2)^{-1} k(z_3, z_1) \\ &= (\mathbf{1}_q - z_2^* z_1)^{-1} (\mathbf{1}_q - z_2^* z_3) (\mathbf{1}_q - z_1^* z_3)^{-1}. \end{aligned}$$

It satisfies the following transformation law

$$(25) \quad T(g(z_1), g(z_2), g(z_3)) = j(g, z_1) T(z_1, z_2, z_3) j(g, z_1)^*$$

for $g \in G$. In particular, we see that $T(\sigma_1, \sigma_2, \sigma_3)$ is well defined on S_{\top}^3 and that the $\text{GL}(q, \mathbf{C})$ -orbit of $T(\sigma_1, \sigma_2, \sigma_3)$ is constant along any G -orbit in S_{\top}^3 .

LEMMA 4.1. *Let $\sigma = \begin{pmatrix} \sigma_p \\ \sigma' \end{pmatrix} \in S$, transverse to ie and $-ie$. Then*

$$(26) \quad T(ie, -ie, \sigma) = \frac{1}{2i} (i\mathbf{1}_q + \sigma_q) (\mathbf{1}_q + i\sigma_q)^{-1}.$$

Proof. This is an easy computation.

PROPOSITION 4.2. *Let $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$. Then*

$$2i T(\sigma_1, \sigma_2, \sigma_3) \in \widetilde{T}_q^{(r)}.$$

Proof. Let us first assume $\sigma_1 = ie, \sigma_2 = -ie, \sigma_3 = \sigma$. Except for the factor $\frac{1}{2i}$, a comparison with (9) shows that $T(ie, -ie, \sigma)$ is the first term of the Cayley transform of σ . More precisely, let $c(\sigma) = \xi = \begin{pmatrix} \xi_q \\ \xi' \end{pmatrix}$. Then we may rewrite (26) as

$$T(ie, -ie, \sigma) = \frac{1}{2i} \xi_q.$$

Now ξ belongs to ${}^c S$, and hence $\frac{1}{2i} (\xi_q - \xi_q^*) = \xi'^* \xi'$. But $\text{rank}(\xi') \leq r$, so $\text{rank}(\xi'^* \xi') \leq r$ and hence ξ_q belongs to $\widetilde{T}_q^{(r)}$. Now the transformation law (25) for the triple ratio implies that for any $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$, $2i T(\sigma_1, \sigma_2, \sigma_3)$ belongs to $\widetilde{T}_q^{(r)}$. \square

THEOREM 4.3. *Let $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) belong to S_{\top}^3 . They belong to the same G -orbit if and only if $T(\sigma_1, \sigma_2, \sigma_3)$ and $T(\tau_1, \tau_2, \tau_3)$ belong to the same $\mathrm{GL}(q, \mathbf{C})$ -orbit.*

Proof. One way is obvious from the transformation law (25) for the triple ratio. For the converse, we assume (as we may) that $\sigma_1 = \tau_1 = ie$ and $\sigma_2 = \tau_2 = -ie$, and set for simplicity $\sigma = \sigma_3$ and $\tau = \tau_3$. Then the assumption implies that $(i\mathbf{1}_q + \sigma_q)(\mathbf{1}_q - i\sigma_q)^{-1}$ and $(i\mathbf{1}_q + \tau_q)(\mathbf{1}_q - i\tau_q)^{-1}$ are in the same $\mathrm{GL}(q, \mathbf{C})$ -orbit. By Lemma 2.3, $c(\sigma)$ and $c(\tau)$ are in the same ${}^c L$ -orbit. So σ and τ are in the same L -orbit. \square

Now to give a description of the invariant in terms of Theorem 3.13, we need to define the analog of the function $\arg \det$. For $z_1 \in D$ and $z_2 \in \bar{D}$, the function $k(z_1, z_2) = (\mathbf{1}_q - z_2^* z_1)^{-1}$ is well defined and belongs to $\mathrm{GL}(q, \mathbf{C})$. So we can extend the definition of T to the set

$$\tilde{D}_{\top} = \{(z_1, z_2, z_3) \mid z_i \in D \cup S, 1 \leq i \leq 3, z_1 \top' z_2, z_2 \top' z_3, z_3 \top' z_1\},$$

where by definition $z \top' w$ is satisfied if z or w belongs to D , and reduces to the condition $z \top w$ if both z and w belong to S . As \tilde{D}_{\top} is stable by $(z_1, z_2, z_3) \mapsto (tz_1, tz_2, tz_3)$ for $0 \leq t \leq 1$, this is a simply connected set. For $z_1 \in D$, $\det T(z_1, z_1, z_1)$ is a positive real number. So there is a well defined continuous determination of the argument of $\det(T(z_1, z_2, z_3))$ on \tilde{D}_{\top} such that it takes the value 0 whenever $z_1 = z_2 = z_3 \in D$. Denote this determination by $\arg \det T(z_1, z_2, z_3)$. It is clearly invariant under the G -action, and so it defines an invariant for the G -orbits.

On the other hand, let

$$S(z_1, z_2, z_3) = T(z_1, z_2, z_3)^{*^{-1}} T(z_1, z_2, z_3)$$

be the angular matrix associated to $T(z_1, z_2, z_3)$.

THEOREM 4.4. *Let $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) belong to S_{\top}^3 . They belong to the same G -orbit if and only if $S(\sigma_1, \sigma_2, \sigma_3)$ and $S(\tau_1, \tau_2, \tau_3)$ are conjugate under $\mathrm{GL}(q, \mathbf{C})$ and $\arg \det T(\sigma_1, \sigma_2, \sigma_3) = \arg \det T(\tau_1, \tau_2, \tau_3)$.*

Proof. This is a direct consequence of Theorem 4.3 and Theorem 3.13.

REMARK 1. Let us consider the case where $q = 1$. The Stiefel manifold is the unit sphere S^{2p-1} in \mathbf{C}^p . The transversality condition $\sigma \top \tau$ just means $\sigma \neq \tau$, as is easily seen from the Cauchy-Schwarz inequality. The triple ratio

is the complex number

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}.$$

The group $\mathrm{GL}(q, \mathbf{C}) \simeq \mathbf{C}^*$ acts on the upper halfplane by $(\lambda, z) \mapsto |\lambda|^2 z$ and so the orbits are described by the argument of the complex number z . So the characteristic invariant in this case is just

$$\arg((1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}).$$

It is equivalent to the invariant θ considered in [KR]. This invariant, almost in our terms, was known to E. Cartan (see [Ca]).

REMARK 2. Let us consider the case where $p = q$. Then the Stiefel manifold is $\mathrm{U}(q)$, and the content of Proposition 4.2 is that for $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}$$

is an invertible skew-Hermitian matrix. The orbits of $\mathrm{GL}(q, \mathbf{C})$ in its action on nondegenerate Hermitian forms are characterized by the signature. So the characteristic invariant as described in Theorem 4.3 in this case reduces to $\mathrm{sgn} iT(\sigma_1, \sigma_2, \sigma_3)$. As concerns Theorem 4.4, notice that the invariant S is trivial (equal to $-\mathbf{1}_q$), so one is only concerned with the invariant $\arg \det T$. The bounded domain D is of tube type and the description of the invariant through the function $\arg \det$ coincides with the approach of this problem in [CØ], where the invariant was introduced under the name of *generalized Maslov index*.

REFERENCES

- [Ca] CARTAN, E. Sur le groupe de la géométrie hypersphérique. *Comment. Math. Helv.* 4 (1932), 158–171.
- [CØ] CLERC, J.-L. and B. ØRSTED. The Maslov index revisited. *Transform. Groups* 6 (2001), 303–320.
- [FK] FARAUT, J. and A. KORÁNYI. *Analysis on Symmetric Cones*. Oxford Mathematical Monographs, Clarendon Press, Oxford, 1994.
- [Fal] FARAUT, J. et al. *Analysis and Geometry on Complex Homogeneous Domains*. Progress in Mathematics 185. Birkhäuser Verlag, Boston, 2000.
- [H] HUA, L.-K. Geometries of matrices, I. Generalizations of von Staudt's theorem. *Trans. Amer. Math. Soc.* 57 (1945), 441–481.
- [KR] KORÁNYI, A. and H. M. REIMANN. The complex cross ratio on the Heisenberg group. *L'Enseign. Math.* (2) 33 (1987), 291–300.