1. Semistable degenerations of K3 -surfaces

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smooth model is the icosahedron.

This paper is organised as follows. In the first section I recall the results on degenerations of $K3$-surfaces, in particular that one can always realise a particularly nice model, the (-1)-form. Section 2 brings as illustration detailed computations for tetrahedra. The results fit in with the general deformation theory, which is treated in the third section, with special emphasis on degenerations in (-1)-form. A short fourth section introduces the combinatorial tools to handle large systems of equations: the definitions of Stanley-Reisner rings and Hodge algebras are reviewed. The final section contains the dodecahedral degenerations.

1. Semistable degenerations of $K3$-surfaces

1.1. The name $K3$ has been explained by André Weil: «en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne $K2$ au Cachemire» [W, p. 546]. He calls any surface a $K3$, if it has the differentiable structure of a smooth quartic surface in $\mathbf{P}^3(\mathbf{C})$. A Kummer surface is a quartic with 16 A_1-singularities. As these singularities admit simultaneous resolution, the minimal resolution of a Kummer surface deforms into a smooth quartic and is therefore a $K3$-surface. A quartic surface X is simply connected, so in particular $b_1(X) = 0$ and has trivial canonical sheaf by the adjunction formula: X is an anti-canonical divisor in $\mathbf{P}^3$. The modern definition of a $K3$-surface: $b_1(X) = 0$ and $K_X = 0$, is equivalent with Weil's definition because all $K3$-surfaces form one connected family.

1.2. Let $f\colon \mathcal{X} \to S \ni 0$ be a proper surjective holomorphic map of a 3-dimensional complex manifold $\mathcal{X}$ to a (germ of a) curve S such that the zero fibre $X = f^{-1}(0)$ is a reduced divisor with (simple) normal crossings; then the degeneration f is called *semistable*.

In the $K3$ case the following holds (see [F-M] for exact references):

1.3. Theorem (Kulikov). *Let $f\colon \mathcal{X} \to S$ be a semistable degeneration of $K3$-surfaces. If all components of $X = f^{-1}(0)$ are Kähler, then there exists a modification $\mathcal{X}'$ of $\mathcal{X}$ such that $K_{\mathcal{X}'} \equiv 0$.*

A degeneration as in the conclusion of the theorem ($K_{\mathcal{X}} \equiv 0$) is called a *Kulikov model.*

1.4. THEOREM (Persson, Kulikov). *Let* $f\colon \mathcal{X} \to S$ *be a Kulikov model of a degeneration of* $K3$*-surfaces with all components of* $X = f^{-1}(0)$ *Kähler. Then either*

(I) X *is smooth, or*

(II) X *is a chain of elliptic ruled components with rational surfaces at the ends and all double curves are smooth elliptic curves, or*

(III) X *consists of rational surfaces meeting along rational curves which form cycles on each component. The dual graph is a triangulation of* S^2.

According to the case division in the theorem one speaks of degenerations of type I, II, or III. Without the Kähler assumption it is not always possible to arrange that $K_{\mathcal{X}} \equiv 0$ [K, N]. Even under the assumption $K_{\mathcal{X}} \equiv 0$ the list becomes longer (see [N, Thm. 2.1]). In particular it is possible that the central fibre contains surfaces of type VII_0. The case that the central fibre contains an Inoue-Hirzebruch surface is relevant for the deformation of cusp singularities [L].

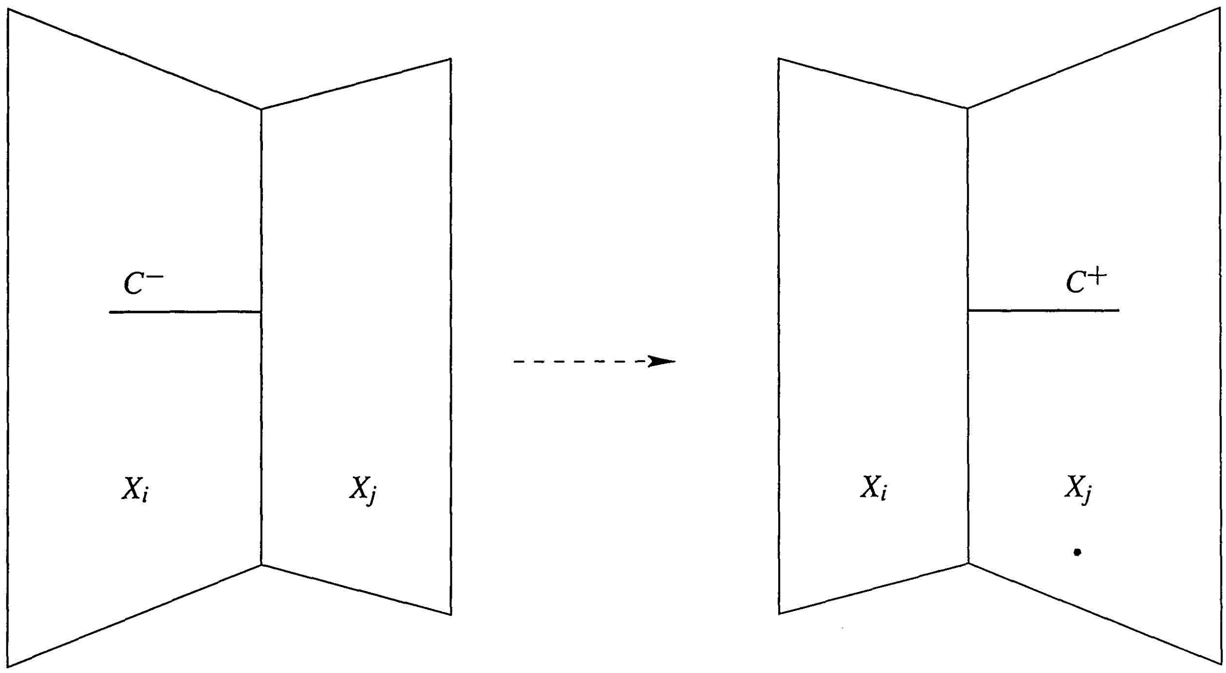

FIGURE 1.1

Elementary modification of type I

A Kulikov model is not unique. The central fibre can be modified with flops. If C^- is a smooth rational curve in X with self intersection -1, lying in a component X_i and intersecting the double curve transversally in one point lying in X_j, then after the flop the curve C^+ lies in X_j. This operation is also called an elementary modification of type I along C^-. An elementary modification of type II is a flop in a curve C^-, which is a component of the double curve and has self intersection -1 on both components X_i, X_j on which it lies. There are two triple points on C^- involving the components X_k and X_l. After the flop C^+ is a double curve lying in the components X_k and X_l. Note that we might lose projectivity by using elementary transformations.

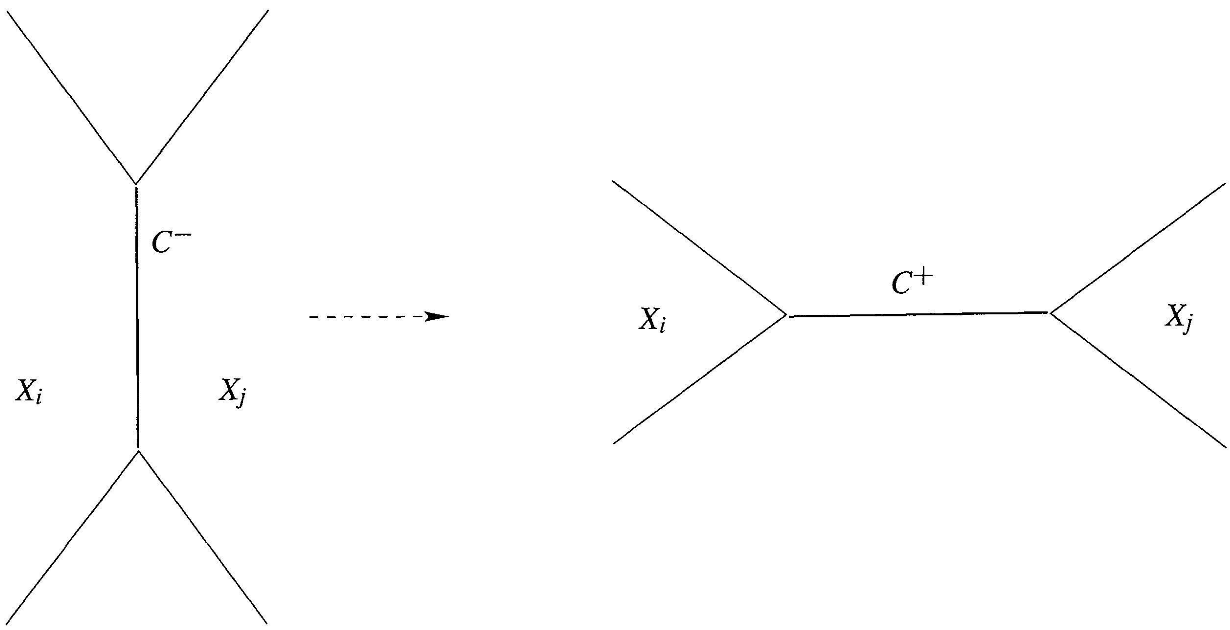

FIGURE 1.2
Elementary modification of type II

1.5. THE MINUS ONE THEOREM [M-M]. *By modifications of type I and II one can achieve that every component of the double curve of the special fibre has self intersection -1 on both components on which it lies.*

1.6. Let $X = \bigcup X_i$ be a normal crossings surface with double locus D. If X is a divisor in a smooth 3-fold M then one can define the infinitesimal normal bundle $\mathcal{O}_D(X)$ as $\mathcal{O}_D(X) = \mathcal{O}_M(X)|_D$. It can be defined independently of M. To this end, let I_{X_i} be the ideal sheaf of X_i in X. It is locally generated by one generator z_i, but is not invertible as z_i is a zero divisor in $\mathcal{O}_X$. However $I_{X_i}|_D$ is locally free [F2, (1.8)]. Following Friedman one makes the following definitions.

1.7. DEFINITION. The *infinitesimal normal bundle* $\mathcal{O}_D(X)$ is the line bundle dual to $\mathcal{O}_D(-X)$, where

$$\mathcal{O}_D(-X) = (I_{X_1}|_D) \otimes_{\mathcal{O}_D} \cdots \otimes_{\mathcal{O}_D} (I_{X_k}|_D).$$

If X is a divisor in M the bundle thus defined is equal to $\mathcal{O}_M(X)|_D$. In particular, if X is a central fibre in a semistable degeneration $\mathcal{X} \to S$, then $\mathcal{O}_{\mathcal{X}}(X) \equiv \mathcal{O}_{\mathcal{X}}$ so $\mathcal{O}_D(X) = \mathcal{O}_D$. This gives a necessary condition for being a central fibre.

1.8. DEFINITION. The normal crossings surface X is *d-semistable* if $\mathcal{O}_D(X) = \mathcal{O}_D$.

A consequence is the triple point formula: let $D_{ij} = X_i \cap X_j$ and denote by $(D_{ij})^2_{X_i}$ the self intersection of D_{ij} on X_i and by T_{ij} the number of triple points on D_{ij}. Then (cf. [P, Cor. 2.4.2])

$$(D_{ij})^2_{X_i} + (D_{ij})^2_{X_j} + T_{ij} = 0\,.$$

1.9. DEFINITION. A compact normal crossings surface is a *d-semistable $K3$-surface of type* III if X is d-semistable, $\omega_X = \mathcal{O}_X$ and each X_i is rational, the double curves $D_i \subset X_i$ are cycles of rational curves and the dual graph triangulates S^2. If the conclusions of the Minus One Theorem hold, that every component of the double curve has self intersection -1 on either component of X on which it lies, the surface X is said to be in *(-1)-form.*

2. TETRAHEDRA

2.1. To realise a tetrahedron we start out with four general planes in 3-space. They do not form a d-semistable $K3$-surface, but the dual graph is a tetrahedron. To write down a degeneration with this special fibre we just take the pencil spanned by $T = x_0x_1x_2x_3$ and a smooth quartic. The symmetry group of the tetrahedron (including reflections) acts if we only take S_4-invariant quartics:

$$Q = a\sigma_1^4 + b\sigma_1^2\sigma_2 + c\sigma_2^2 + d\sigma_1\sigma_3\,,$$

where the σ_i are the elementary symmetric functions in the four variables x_i and a, b, c and d are constants.

To obtain a family $f\colon \mathcal{X} \to S$ one has to blow up the base locus of the pencil. This can be done in several ways. Blowing up $T = Q = 0$ gives a total space which is singular, with in general 24 ordinary double points coming from the 24 intersection points of Q with the double curve of the tetrahedron T. Arguably, this is the nicest model, and the best one can hope for in view of the theory of minimal models of 3-folds. A smooth model is