4. HODGE ALGEBRAS

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3.15. A nice construction for studying the homology of the general fibre is given by [A'C]. Let $\sigma_i\colon \mathcal{Z}_i \to \mathcal{X}$ be the oriented real blow-up of $X_i \subset \mathcal{X}$. This is a manifold with boundary, whose boundary $\partial\mathcal{Z}_i = \sigma_i^{-1}(X_i)$ is isomorphic to the boundary of a tubular neighbourhood of X_i in $\mathcal{X}$. The fibred product $\sigma\colon \mathcal{Z} \to \mathcal{X}$ of the σ_i is a manifold with corners. Its boundary $\mathcal{N} := \partial\mathcal{Z}$ comes with a map to X. It also fibres over S^1 : the composed map $\mathcal{Z} \to \mathcal{X} \to S \ni 0$ extends to a map from $\mathcal{Z}$ to the real oriented blow-up of S in 0 (polar coordinates !). A fibre of $\mathcal{N} \to S^1$ is then a topological model of the general fibre.

This model is not sufficient to describe the monodromy. One has first to replace X by the geometric realisation of the simplicial object $X^{[\cdot]}$: one replaces each double point by an interval, and each triple point by a 2-simplex. A final fibred product then gives the new model. For details see [A'C, §2].

4. Hodge algebras

4.1. Stanley–Reisner rings. Let Δ be a simplicial complex with set of vertices $V = \{v_1, \ldots, v_n\}$. A monomial on V is an element of $\mathbf{N}^V$. Each subset of V determines a monomial on V by its characteristic function. The support of a monomial $M\colon V \to \mathbf{N}$ is the set $\operatorname{supp} M = \{v \in V \mid M(v) \neq 0\}$. The set Σ_Δ of monomials whose support is not a face is an ideal, generated by the monomials corresponding to minimal non-simplices.

Given a ring R and an injection $\phi\colon V \to R$ we can associate to each monomial M on V the element $\phi(M) = \prod_{v\in V} \phi(v)^{M(v)} \in R$. We will usually identify V and $\phi(V)$ and write $M \in R$ for $\phi(M)$. This applies in particular to the polynomial ring $K[V]$ over a field K. The ideal Σ_Δ gives rise to the Stanley-Reisner ideal $I_\Delta \subset K[V]$. The *Stanley-Reisner ring* is $A_\Delta = K[V]/I_\Delta$.

Deformations of Stanley-Reisner rings are studied in [A-C].

4.2. Example. Let Δ be an octahedron. We map the set of vertices to $\mathbf{C}[x_1, \ldots, x_6]$ such that opposite vertices correspond to variables with index sum 7.

The Stanley-Reisner ring is minimally generated by the three monomials $x_i x_{7-i}$. The spaces smoothes to a $K3$-surface, the complete intersection of three general quadrics. A general 1-parameter deformation is not semi-stable, because the total space has singularities at the six quadruple points of the special fibre.

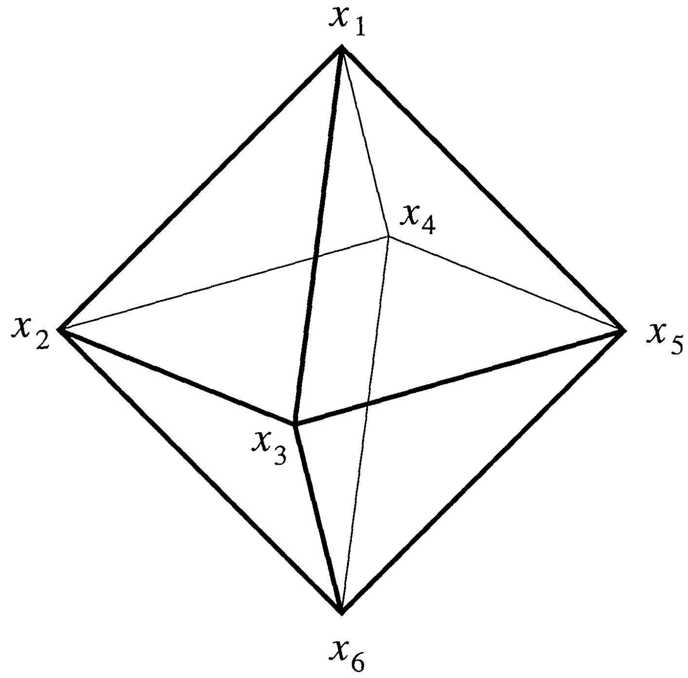

FIGURE 4.1
Octahedron

4.3. REMARK. To get an octahedron as dual graph we need the incidence relations of a cube. The toric variety associated to a cube is $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. The general anticanonical divisor is a smooth $K3$, whereas the complement of the torus is the unions of six quadrics (each of the form $\mathbf{P}^1 \times \mathbf{P}^1$) with dual graph the octahedron. A small resolution of a general pencil yields a semistable degeneration.

4.4. DEFINITION [D-E-P]. Let H be a finite partially ordered set and Σ an ideal of monomials on H. Let K be a commutative ring and R a commutative K-algebra. Suppose an injection $\phi\colon H \to R$ is given. Then R is a *Hodge algebra* (or algebra with straightening law) *governed by* Σ if

H–1 R is a free K-module admitting the set of monomials not in Σ as basis,

H–2 for each generator M of Σ in the unique expression

$$(*) \qquad M = \sum_{N \notin \Sigma} k_{M,N} N, \qquad k_{M,N} \in K,$$

guaranteed by H–1, for each $x \in H$ dividing M and each $N \notin \Sigma$ with $k_{M,N} \neq 0$ there is a $y_{M,N}$ dividing N and satisfying $y_{M,N} < x$.

The relations $(*)$ are called *straightening relations* for R.

4.5. If R is graded and the elements of $\phi(H)$ are homogeneous the straightening relations give a presentation for R [D-E-P, p. 15].

We note that R is a deformation of the discrete Hodge algebra governed by Σ, whose ideal is generated by the monomials M.

4.6. EXAMPLE. The equations of the tetrahedron of degree 12 of 2.2 are straightening relations. We take Σ as Stanley-Reisner ideal Σ_Δ, where Δ is the stellation of the tetrahedron: in each top-dimensional face we take an additional vertex, which is joined to all vertices on the face. The partial order on the set of vertices is obtained by declaring the new vertices to be smaller. The discrete Hodge algebra then has equations x_ix_j, x_iy_j and $y_jy_ky_l$.

5. THE DODECAHEDRON

5.1. To get an icosahedron as dual graph we need the incidence relations of a dodecahedron. Each side should be a rational surface and the intersection with the other surfaces should have a pentagon as dual graph. A pentagon occurs as hyperplane section of a Del Pezzo surface of degree 5. So we can realise our dodecahedron by gluing together 12 Del Pezzo surfaces.

We first describe the Del Pezzo surfaces. Each of those is an extension of its pentagonal hyperplane section. Its coordinate ring can be obtained as Stanley-Reisner ring of a pentagon as 1-dimensional simplicial complex. Introducing variables y_i, we get the equations $y_{i-1}y_{i+1}$. With an extra variable x the Del Pezzo surface has equations

$$y_{i-1}y_{i+1} - xy_i - x^2 .$$

These are the Pfaffians of the matrix

$$\begin{pmatrix} 0 & y_1 & x & -x & -y_5 \\ -y_1 & 0 & y_2 & x & -x \\ -x & -y_2 & 0 & y_3 & x \\ x & -x & -y_3 & 0 & y_4 \\ y_5 & x & -x & -y_4 & 0 \end{pmatrix} .$$

We can check that this is indeed a smooth Del Pezzo of degree 5 by giving an explicit birational map from $\mathbf{P}^2$, which blows up four points, see Figure 5.1.

To the variable x corresponds a new vertex at the centre of the pentagon. By joining it to all other vertices we obtain a 2-dimensional simplicial complex, and the homogeneous coordinate ring of the Del Pezzo surface is a graded Hodge algebra governed by the Stanley-Reisner ideal of the complex: to satisfy H–2 we take x to be less than all y_i.