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THE COSET WEIGHT DISTRIBUTIONS OF CERTAIN BCH CODES AND A FAMILY OF CURVES

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INTRODUCTION

Many problems in coding theory are related to the problem of determining the distribution of the number of rational points in a family of algebraic curves defined over a finite field. Usually, these problems are very hard and a complete answer is often out of reach.

In the present paper we consider the problem of the weight distributions of the cosets of certain BCH codes. This problem turns out to be equivalent to the determination of the distribution of the number of points in a family of curves with a large symmetry group. The symmetry allows us to analyze closely the nature of these curves and in this way we are able to extend considerably our control over the coset weight distribution compared with earlier results.

For a binary linear code C of length n the weight distributions of the cosets of C in $\mathbf{F}_2^n$ are important invariants of the code. They determine for example the probability of a decoding error when using C. However, the coset weight distribution problem is solved for very few types of codes.

In [C-Z] Charpin and Zinoviev study the weight distributions of the cosets of the binary 3-error-correcting BCH code of length $n = 2^m - 1$ with m odd. We denote this code by $BCH(3)$.

Let $\mathbf{F}_q$ be a finite field of cardinality $q = 2^m$ and let α be a generator of the multiplicative group $\mathbf{F}_q^*$. The matrix

$$H = \begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ 1 & \alpha^3 & \alpha^6 & \dots & \alpha^{3(n-1)} \\ 1 & \alpha^5 & \alpha^{10} & \dots & \alpha^{5(n-1)} \end{pmatrix}$$

is a parity check matrix defined over $\mathbf{F}_q$ of $BCH(3)$. This means that

$$BCH(3) = \{c = (c_0, \dots, c_{n-1}) \in \mathbf{F}_2^n : Hc^t = 0\}.$$

It was shown in [C-Z] that the coset weight distribution problem for $BCH(3)$ comes down to the same problem for the extended code $\widehat{BCH(3)}$ with parity check matrix

$$\widehat{H} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} & 0 \\ 1 & \alpha^3 & \alpha^6 & \dots & \alpha^{3(n-1)} & 0 \\ 1 & \alpha^5 & \alpha^{10} & \dots & \alpha^{5(n-1)} & 0 \end{pmatrix}$$

A coset $\widehat{D}$ of $\widehat{BCH(3)}$ in $\mathbf{F}_2^{n+1}$ is characterized by the syndrome $s(\widehat{D}) = \widehat{H}x^t \in \mathbf{F}_q^4$, where x is a representative of $\widehat{D}$. The weight of $\widehat{D}$ is the minimum weight of the vectors in $\widehat{D}$. Here the weight of a vector is the number of its non-zero entries.

Charpin and Zinoviev then show that the weight distribution problem for the cosets of $\widehat{BCH(3)}$ of length 2^m with m odd can be solved as soon as the weight distributions of the cosets $\widehat{D}_4$ of weight 4 with syndrome $s(\widehat{D}_4) = (0, 1, A, B)$ are determined.

From [C-Z] we recall: *The weight distribution of a coset $\widehat{D}_4$ is determined by the number $N(A,B)$ of vectors of weight 4 in $\widehat{D}_4$.*

Via the matrix $\widehat{H}$ this leads to the system of equations in four variables in $\mathbf{F}_{q=2^m}$:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1, \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 &= A, \\ x_1^5 + x_2^5 + x_3^5 + x_4^5 &= B, \end{aligned} \tag{1}$$

and $N(A,B)$ is the number of S_4-orbits of solutions of (1) with distinct $x_i \in \mathbf{F}_q$. In particular the number of values of $N(A,B) > 0$ equals the number of different coset weight distributions of cosets of type $\widehat{D}_4$. Note that since the set of solutions of (1) is invariant under translation over $(1,1,1,1)$ the quantity $N(A,B)$ is even.

In this paper we shall show that by analyzing carefully the curves defined by (1) we can determine good upper and lower bounds for the pivotal quantity $N(A,B)$. The bounds are obtained by dissecting the Jacobian variety of the curves in our family in isogeny factors of dimension 1 and 2. This yields restrictions on the traces of Frobenius. The splitting of the Jacobian is a corollary from a very effective description of the curves defined by (1) as fibre products over $\mathbf{P}^1$ of three elliptic curves.

We show that for odd m the $N(A,B)$ lie in an explicit interval of length $\sim 1.57\sqrt{q}$, cf. [C-Z], where the interval is $\sim q/4$. Moreover, we argue that on statistical grounds one may expect that almost all $N(A,B)$ lie in an explicit interval of length $\sim 0.9\sqrt{q}$. We then give numerical results that confirm strongly these heuristics and extend the table of $BCH(3)$ codes with known coset weight distribution.

For an introduction to the theory of codes we refer to [vL] and for a general introduction to curves over finite fields to [S]. The reader can find basic facts about Jacobians in the survey paper [Mi] and a general introduction to curves and their Jacobians in [Mu].

§1. A FAMILY OF CURVES

We consider the algebraic curve $C' = C'_{A,B}$ in $\mathbf{P}^4$ given by the equations

$$s_1 = x_0 , \quad s_3 = Ax_0^3 , \quad s_5 = Bx_0^5 , \tag{2}$$

where s_j is the j-th power sum $\sum_{i=1}^4 x_i^j$ in the variables $x_1, \ldots, x_4$. Let σ_j denote the j-th elementary symmetric function in $x_1, \ldots, x_4$. If we apply Newton's formulas for power sums we find

$$\begin{aligned} s_1 + x_0 &= \sigma_1 + x_0 = 0 , \\ s_3 + Ax_0^3 &= (A+1)x_0^3 + \sigma_2 x_0 + \sigma_3 = 0 , \\ s_5 + Bx_0^5 &= x_0 \left((B+A)x_0^4 + (A+1)\sigma_2 x_0^2 + \sigma_4\right) = 0 . \end{aligned}$$

This implies that the curve C' consists of the three lines in the hyperplane $x_0 = 0$ given by

$$x_i + x_j = x_k + x_l = 0 , \quad \text{with} \quad \{i,j,k,l\} = \{1,2,3,4\} , \tag{3}$$

and a curve $C = C_{A,B}$ given by

$$\begin{aligned} \sigma_1 &= x_0 , \\ \sigma_3 &= (A+1)x_0^3 + \sigma_2 x_0 , \\ \sigma_4 &= (B+A)x_0^4 + (A+1)\sigma_2 x_0^2 . \end{aligned} \tag{4}$$

The symmetric group S_4 operates on C' and on C by permuting the coordinates $x_1, \ldots, x_4$. Moreover, there is an involution τ acting on C via

$$(x_0 : x_1 : \ldots : x_4) \mapsto (x_0 : x_1 + x_0 : \ldots : x_4 + x_0) .$$

This involution commutes with the elements of S_4 and this gives rise to a group of 48 automorphisms of C.

We introduce the invariant

$$\lambda := B + A^2 + A + 1 \quad (\in \mathbf{F}_q).$$

In the following lemma and the rest of this section we shall work over an algebraic closure of $\mathbf{F}_q$.

(1.1) LEMMA.

i) *If* $\lambda \neq 0$ *then* C *has six ordinary double points, namely the points of the* S_4*-orbit of* $(0:1:1:0:0)$ *and no other singularities.*

ii) *If* $\lambda = 0$ *the curve* C *consists of* 12 *lines.*

Proof. The Jacobian matrix of (2) is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ Ax_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ Bx_0^4 & x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{pmatrix}.$$

If the rank of this matrix is ≤ 2 for a point with coordinates $(x_0 : \ldots : x_4)$ then there exist α, β, γ with α, β, γ not all zero such that $\alpha + \beta x_i^2 + \gamma x_i^4 = 0$ for $i = 1, \ldots, 4$. Hence the coordinates x_i with $i = 1, \ldots, 4$ of a singular point of C can assume at most 2 different values and taking into account the equation $s_1 = x_0$ it follows that a singular point of C is in the S_4-orbit of a point of the form $(a : 1 : 1 : 1 : a+1)$ or of the form $(0 : 1 : 1 : a : a)$ for some value of a. In the latter case we get from (4) that $a = 0$ and we find 6 singular points in the orbit of $(0 : 1 : 1 : 0 : 0)$. In the former case it follows from (4) that a satisfies

(5) $\quad (A+1)a^3 + a^2 + a = 0, \quad$ and $\quad (B+A)a^4 + (A+1)a^3 + a + 1 = 0.$

Hence $a \neq 0$ and (5) is equivalent to

$$(6) \qquad \begin{aligned} (A+1)a^2 + a + 1 &= 0, \\ (B+A)a^2 + (A+1)a + (A+1) &= 0. \end{aligned}$$

The resultant of (6) equals $(B + A^2 + A + 1)^2$, hence (6) has a solution if and only if $\lambda = B + A^2 + A + 1$ vanishes. In that case the Jacobian matrix has rank 2 for the solutions of (6).

So if $\lambda \neq 0$ the curve C has six singular points, namely the S_4-orbit of $(0 : 1 : 1 : 0 : 0)$. For the local structure near $(0 : 1 : 1 : 0 : 0)$ we eliminate x_0 from (2) and find that the curve C' in $\mathbf{P}^3$ is given by

$$s_3 = As_1^3, \quad s_5 = Bs_1^5.$$

Upon taking affine coordinates $\xi_1 = (x_1+x_2)/x_1$, $\xi_2 = x_3/x_1$, $\xi_3 = x_4/x_1$ we find the equations

$$\xi_1 + \xi_1^2 + \xi_1^3 + \xi_2^3 + \xi_3^3 = A(\xi_1 + \xi_2 + \xi_3)^3,$$
$$\xi_1 + \xi_1^4 + \xi_1^5 + \xi_2^5 + \xi_3^5 = B(\xi_1 + \xi_2 + \xi_3)^5.$$

This shows that ξ_1 lies in m^3, with m the maximal ideal of $(0,0,0)$ in $\mathbf{A}^3$ and defines the tangent plane at $(0,0,0)$ to the cubic surface S given by the cubic equation. Moreover, this is also the lowest order term of the quintic equation. Therefore, locally near the origin C' is given by

$$\xi_1 = 0, \quad (\xi_2+\xi_3)(\xi_2\xi_3 + (A+1)(\xi_2+\xi_3)^2) = 0. \tag{7}$$

which shows that C' has a triple point and C has a node at this point.

If $\lambda = 0$ and a satisfies $(A+1)a^2 + a + 1 = 0$ then a is a solution of (6) and the S_4-orbit of points of the form $(a : x : x : 1 : a+1)$ with arbitrary x is on C. So the equations

$$x_i + x_j = 0, \quad (a+1)x_k + x_l = 0 \quad \text{with} \quad \{i,j,k,l\} = \{1,2,3,4\}$$

define a line on C and this gives 12 lines on C. Since C has degree 12 the curve C decomposes as the union of 12 lines. This proves ii).

REMARK. It follows from the preceding proof that for $\lambda \neq 0$ points on C for which $x_1, \dots, x_4$ are not all distinct lie on one of the lines (3).

(1.2) PROPOSITION. *If $\lambda \neq 0$ then C is irreducible.*

Proof. Suppose that $C = \sum_{i=1}^{\ell} C_i$ is a sum of irreducible components C_i with $\ell \geq 2$. Since C is connected at least one of the singular points is an intersection point of two distinct components C_i. By the S_4-symmetry then each of the six singular points is an intersection point of two different components. This implies that the components C_i are non-singular. Since the permutation (34) interchanges the two branches of C in $(0:1:1:0:0)$ (cf. (7)) the group S_4 acts transitively on the branches through a singular point, so S_4 acts transitively on the set of components.

Let S be the smooth cubic surface in $\mathbf{P}^4$ given by the equations $s_1 = x_0$, $s_3 = Ax_0^3$. On S the curve C is linearly equivalent to $4H$ with H the hyperplane section of S. Now the intersection number HC_i equals the intersection number with the hyperplane $x_0 = 0$, i.e. the intersection number of C_i with the three lines (3), and since the intersection is transversal HC_i

equals the number of singular points of C on C_i. Put $r = 12/\ell$. Then by the symmetry we have $HC_i = r$. On the other hand, the adjunction formula

$$C_i^2 + K_S C_i = C_i^2 - HC_i = C_i^2 - r = 2g(C_i) - 2\,,$$

where K_S is the canonical divisor of S, and the identity

$$4r = 4HC_i = CC_i = C_i^2 + \sum_{j \neq i} C_i C_j = C_i^2 + r$$

imply $C_i^2 = 3r$ and $g(C_i) = r + 1$. In particular, C_i cannot be contained in a hyperplane and spans $\mathbf{P}^3$. Clifford's theorem applied to the hyperplane section $H|C_i$ of C_i says that $h^0(H|C_i) \leq r/2 + 1$, hence $r \geq 6$. Then $\ell = 2$ and we have two components. Again, by Clifford, these curves must be hyperelliptic and the linear system $H|C_i$ is $3g_2^1$. But since $3g_2^1$ is contained in the canonical system $|K_{C_i}|$ this factors through the hyperelliptic involution, which contradicts the fact that C_i is embedded in $\mathbf{P}^3$ as a non-rational curve. This proves that C is irreducible.

(1.3) COROLLARY. *If $\lambda \neq 0$ the normalization $\widetilde{C}$ of C is an irreducible smooth curve of genus* 13.

Proof. On the cubic surface S we have $(C + K_S)C = (4-1)HC = 36$. This implies that for $\widetilde{C}$ we have $2g(\widetilde{C}) - 2 = 36 - 12 = 24$. $\square$

§2. DISSECTING THE JACOBIAN

For the sake of convenience when we refer to a curve in the sequel we shall always mean the normalization of (a completion of) that curve. In particular, by the genus we mean the geometric genus of the curve and if we speak of the number of rational points we mean the number of rational points of the normalization. Note that an absolutely irreducible curve D has a unique complete non-singular model D' obtained by normalizing any completion of the curve. Any automorphism of the curve D defines uniquely an automorphism of the normalization D'.

We now analyze the absolutely irreducible curve $C = C_{A,B}$ for $\lambda \neq 0$ in more detail in order to decompose its Jacobian.

Let $H \subset \mathrm{Aut}(C)$ be the subgroup generated by the two permutations (12) and (34) and the involution τ. Then H is abelian of order 8 and isomorphic to $(\mathbf{Z}/2\mathbf{Z})^3$. Consider the following diagram of degree 2 coverings of curves

$$\begin{tikzcd}
 & C \arrow[dl] \arrow[d] \arrow[dr] & \\
C/(12) \arrow[d] \arrow[dr] & C/(12)(34) \arrow[d] & C/(34) \arrow[dl] \arrow[d] \\
C/\langle(12),\tau\rangle \arrow[dr] & C/\langle(12),(34)\rangle \arrow[d] & C/\langle(34),\tau\rangle \arrow[dl] \\
 & C/H &
\end{tikzcd}$$

Let $u=(x_3+x_4)/(x_1+x_2)$. This is a H-invariant rational function on C, hence defines a rational function on C/H.

(2.1) PROPOSITION.

i) *The function* u *gives an isomorphism* $C/H \cong \mathbf{P}^1$.

ii) *The curve* $C/\langle(12),(34)\rangle$ *is a curve of genus* 1 *given by*

$$y^2+y=\lambda u+\lambda/u+(A+1).$$

iii) *The curve* $C/\langle(12),\tau)\rangle$ *is a curve of genus* 2 *given by*

$$y^2+y=\lambda/u^3+\lambda u.$$

iv) *The curve* $C/\langle(34),\tau\rangle$ *is a curve of genus* 2 *given by*

$$y^2+y=\lambda u^3+\lambda/u.$$

Proof. The divisor of u on C is of the form $H_{34}C-H_{12}C$, where H_{ij} is the hyperplane given by $x_i+x_j=0$. Since both these hyperplanes contain the line $x_1+x_2=x_3+x_4=0$ which intersects C in a divisor of degree 4 it follows that the divisor of u can be written as a difference of two divisors of degree $12-4=8$. Moreover, these divisors are invariant under the action of H. This implies that on C/H the function u defines a non-constant function with a single zero and a single pole. Therefore u defines an isomorphism $C/H \cong \mathbf{P}^1$. This proves i).

We now prove ii). Working with the affine equations (set $x_0=1$)

$$\sigma_1=1, \quad \sigma_3=A+1+\sigma_2, \quad \sigma_4=B+A+(A+1)\sigma_2,$$

we can write $u=(x_3+x_4)/(x_1+x_2)=1+1/(x_1+x_2)$, i.e.

$$x_1+x_2=1/(u+1) \quad \text{and} \quad x_3+x_4=u/(u+1).$$

We put $v := x_1x_2$ and $w := x_3x_4$. These functions are invariant under (12) and (34), but not under τ. Using

$$\sigma_2 = x_1x_2 + x_3x_4 + (x_1 + x_2)(x_3 + x_4) = v + w + u/(u + 1)^2\,,$$

$$\sigma_3 = x_1x_2(x_3 + x_4) + (x_1 + x_2)x_3x_4 = (uv + w)/(u + 1)\,,$$

the equation $\sigma_3 = A + 1 + \sigma_2$ implies

$$A(u + 1)^2 + (u + 1)(v + uw) + u^2 + u + 1 = 0\,, \tag{8}$$

while the equation $\sigma_4 = B + A + (A + 1)\sigma_2$ yields

$$B + A + (A + 1)(v + w + u/(u + 1)^2) + vw = 0\,. \tag{9}$$

Elimination of w from (8) and (9) yields the equation

$$(u + 1)^2v^2 + u(u + 1)v = \lambda u^3 + \lambda u + (A + 1)u + (A + 1)^2u^2 + (A + 1)^2\,.$$

Dividing by u^2 and and replacing $(u + 1)v/u$ by η (i.e. $\eta = x_1x_2/(x_3 + x_4)$) gives

$$\eta^2 + \eta = \lambda u + \lambda/u + (A + 1)/u + (A + 1)^2/u^2 + (A + 1)^2 \tag{10}$$

and this is, via $y = \eta + (A + 1)/u + (A + 1)$, clearly $\mathbf{F}_q$-isomorphic to

$$y^2 + y = \lambda u + \lambda/u + A + 1\,.$$

Since η is invariant under (12) and (34), but not under τ, the equation (10) describes the degree 2 cover $C/\langle(12),(34)\rangle$ of C/H.

For iii) we remark that the function field extension of $C/\langle(12),\tau\rangle$ over C/H is generated by the function $z = x_3 + x_1x_2/(x_3 + x_4)$. Then $z + z^{(34)} = x_3 + x_4 = u/(u + 1)$. Moreover,

$$\begin{aligned} z \cdot z^{(34)} &= x_3x_4 + x_1x_2 + (x_1x_2)^2/(x_3 + x_4)^2 \\ &= w + v + \eta^2 \\ &= A(u + 1)/u + 1 + 1/u(u + 1) + \eta + \eta^2\,, \end{aligned}$$

where we used $w = A(u+1)/u+1+1/u(u+1)+v/u$ obtained from (8) and $v + v/u = \eta$. By (10) this implies that z satisfies the equation

$$z^2 + \frac{u}{u+1}z = \frac{\lambda(u^4 + u^3) + (A^2 + A)u^3 + \lambda(u^2 + u) + Au^2 + A^2 + 1}{u^2(u + 1)}\,.$$

Dividing by $(u/(u + 1))^2$ and replacing $(u + 1)z/u$ by ζ gives the equation

$$\zeta^2 + \zeta = \lambda u + \lambda/u^3 + (A^2 + A) + 1/u + 1/u^4 + A/u^2 + A^2/u^4\,.$$

Via $\zeta \mapsto \zeta + A + 1/u + (A + 1)/u^2$ we get the $\mathbf{F}_q$-isomorphic curve

$$\zeta^2 + \zeta = \lambda u + \lambda/u^3\,.$$

Part iv) is now obtained by applying the permutation (13)(24). This changes u into u^{-1} and proves the result.

(2.2) THEOREM. *The normalization of the curve C is the normalization of the fibre product over $\mathbf{P}^1$ with affine coordinate x of the three hyperelliptic curves given by*

$$y^2 + y = \lambda x^3 + \lambda/x,$$
$$y^2 + y = \lambda/x^3 + \lambda x,$$
$$y^2 + y = \lambda x + \lambda/x + A + 1.$$

Proof. This follows directly from the diagram and the preceding proposition.

Note that equivalently, C is the fibre product of the three curves C_{f_i} of genus 1 with affine equation $y^2 + y = f_i$, where f_i for $i = 1, 2, 3$ is given by

$$\begin{aligned} f_1 &= \lambda x^3 + \lambda x + A + 1, \\ f_2 &= \lambda/x^3 + \lambda/x + A + 1, \\ f_3 &= \lambda x + \lambda/x + A + 1, \end{aligned} \tag{11}$$

since f_1, f_2 and f_3 generate the same space of functions as the right hand sides in the theorem. This description allows us to dissect the Jacobian of C.

(2.3) THEOREM. *The Jacobian of $C_{A,B}$ decomposes up to isogeny over $\mathbf{F}_q$ as a product of five supersingular elliptic curves, two ordinary elliptic curves and three 2-dimensional factors of 2-rank 1.*

Proof. From the description of $C = C_{A,B}$ as a fibre product we see that $\mathrm{Jac}(C)$ decomposes as a product of seven factors: three elliptic curves $\mathrm{Jac}(C_{f_i})$, two 2-dimensional factors $\mathrm{Jac}(C_{f_1+f_3})$, $\mathrm{Jac}(C_{f_2+f_3})$, and two 3-dimensional factors $\mathrm{Jac}(C_{f_1+f_2})$ and $\mathrm{Jac}(C_{f_1+f_2+f_3})$. The 2-rank of $\mathrm{Jac}(C_{f_i})$ is 0 for $i = 1, 2$ and 1 for $i = 3$. The 2-ranks of $\mathrm{Jac}(C_{f_1+f_3})$ and $\mathrm{Jac}(C_{f_2+f_3})$ are 1 since these hyperelliptic curves have two Weierstrass points.

The curve $C_{f_1+f_2+f_3}$ is a curve of genus 3 defined by $y^2 + y = \lambda(x^3 + 1/x^3) + A + 1$ with automorphisms

$$\rho\colon (x, y) \mapsto (1/x, y), \quad \sigma\colon (x, y) \mapsto (x, y + 1).$$

The quotient of $C_{f_1+f_2+f_3}$ under ρ is the supersingular elliptic curve given by $y^2+y = \lambda(z^3+z)+A+1$ with $z = x+1/x$. Moreover, the curve $C_{f_1+f_2+f_3}$ admits a non-constant map to the ordinary elliptic curve $y^2+y = \lambda(w+1/w)+A+1$ via $w = x^3$. So by Poincaré's complete reducibility theorem the Jacobian

$\mathrm{Jac}(C_{f_1+f_2+f_3})$ splits up to isogeny into a product of three elliptic curves and has 2-rank 1 since it has 2 ramification points.

Similarly, the quotient of $C_{f_1+f_2}$ by the automorphism ρ is the supersingular elliptic curve $y^2+y=\lambda z^3$, while the quotient under $\rho\sigma$ is a curve of genus 2 of 2-rank 1 defined by the equation $y^2+y=\lambda z^3+1/z$. Collecting these results we obtain the theorem.

For a smooth absolutely irreducible complete curve X defined over a field $\mathbf{F}_q$ we shall denote the trace of Frobenius by $t(X)$, i.e. $t(X)=q+1-\#X(\mathbf{F}_q)$, where $\#X(\mathbf{F}_q)$ is the number of $\mathbf{F}_q$-rational points of X.

(2.4) COROLLARY. *For $q=2^m$ with m odd the trace of Frobenius of $C_{A,B}$ equals $2t(C_{f_1})+2t(C_{f_3})+2t(C_{f_1+f_3})+t(C_{g_\lambda})$, where C_{g_λ} is the curve given by $y^2+y=g_\lambda$ with $g_\lambda=\lambda x^3+1/x$.*

Proof. The curves C_{f_1} and C_{f_2} are isomorphic via $x\mapsto 1/x$, so have the same trace of Frobenius. Since for $q=2^m$ with m odd the map $x\mapsto x^3$ is a bijection on $\mathbf{F}_q$, the curve $C_{f_1+f_2+f_3}$ given by $y^2+y=\lambda(x^3+1/x^3)+A+1$ and the ordinary factor of its Jacobian given by $y^2+y=\lambda(w+1/w)+A+1$ have the same trace of Frobenius, and this is $t(C_{f_3})$. Moreover, since $C_{f_1+f_3}$ and $C_{f_2+f_3}$ are isomorphic, we have $t(C_{f_1+f_3})=t(C_{f_2+f_3})$. Similarly, the supersingular component of $\mathrm{Jac}(C_{f_1+f_2})$ given by $y^2+y=\lambda z^3$ has the same trace of Frobenius as the rational curve $y^2+y=\lambda z$, i.e. 0. Therefore, the trace $t(C_{f_1+f_2})$ equals the trace of the genus 2 quotient $C_{f_1+f_2}/\rho\sigma$, and this is the curve $y^2+y=g_\lambda$.

We can interpret and augment the results obtained using the involution τ. The involution τ acts without fixed points on the normalization of C, hence by the Hurwitz-Zeuthen formula the genus of the quotient curve C/τ is 7. The Jacobian $\mathrm{Jac}(C)$ decomposes up to an isogeny

$$\mathrm{Jac}(C)\sim \mathrm{Jac}(C/\tau)\times P,$$

where P is the Prym variety of $C\to C/\tau$, i.e. the identity component of the norm map $\mathrm{Nm}\colon \mathrm{Jac}(C)\to\mathrm{Jac}(C/\tau)$. Since the curves $C/\langle(12),\tau\rangle=C_{f_2+f_3}$, $C/\langle(34),\tau\rangle=C_{f_1+f_3}$ and $C/\langle(12)(34),\tau\rangle=C_{f_1+f_2}$ are quotients of C/τ and the fibre product $C_{f_1+f_3}\times_{\mathbf{P}^1}C_{f_2+f_3}$ has genus 7 it follows readily that

$$C/\tau\cong C_{f_1+f_3}\times_{\mathbf{P}^1}C_{f_2+f_3}.$$

Note that the substitution $x\mapsto x/\lambda$ yields an isomorphism $C_{g_{\lambda^4}}\cong C_{f_1+f_3}$.

(2.5) PROPOSITION. *Up to isogeny over* $\mathbf{F}_{q=2^m}$ *we have the splitting*

$$\mathrm{Jac}(C/\tau) \sim \mathrm{Jac}(C_{g_{\lambda^4}})^2 \times \mathrm{Jac}(C_{g_\lambda}) \times E\,,$$

where C_{g_λ} *is as in (2.4) and* E *is the elliptic curve* $y^2+y=\lambda z^3$. *The Prym variety* P *is isogenous to a product of six elliptic curves:*

$$P \sim \mathrm{Jac}(C_{f_1})^2 \times \mathrm{Jac}(C_{f_3})^2 \times P'\,,$$

where P' *is a supersingular abelian surface whose trace of Frobenius* $t(P')$ *over* $\mathbf{F}_q$ *satisfies*

$$t(P') = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ -2(q-1)+2t(C_{f_3}) & \text{if } m \text{ is even.} \end{cases}$$

Proof. The splitting of $\mathrm{Jac}(C/\tau)$ follows directly from the description of C/τ as a fibre product and the splitting $\mathrm{Jac}(C_{f_1+f_2}) \sim \mathrm{Jac}(C_{g_\lambda}) \times E$ as obtained in (2.4). Furthermore, using Theorem (2.3) we see that

$$P \sim \mathrm{Jac}(C_{f_1}) \times \mathrm{Jac}(C_{f_2}) \times \mathrm{Jac}(C_{f_3}) \times \mathrm{Jac}(C_{f_1+f_2+f_3})\,.$$

We know $\mathrm{Jac}(C_{f_1}) \cong \mathrm{Jac}(C_{f_2})$ and that $\mathrm{Jac}(C_{f_1+f_2+f_3})$ splits up to isogeny as $\mathrm{Jac}(C_{f_3})$ and a 2-dimensional factor P' which is supersingular and up to isogeny a product of two elliptic curves. Using the map $x \mapsto w = x^3$ we see that $\#C_{f_1+f_2+f_3}(\mathbf{F}_q) = \#C_{f_3}(\mathbf{F}_q)$ if m is odd which implies $t(P')=0$, while for m even

$$\#C_{f_1+f_2+f_3}(\mathbf{F}_q) - 2 = 3(\#C_{f_3}(\mathbf{F}_q) - 2)\,.$$

This implies

$$t(C_{f_1+f_2+f_3}) - t(C_{f_3}) = -2(q-1) + 2t(C_{f_3})$$

and hence $t(P') = -2(q-1)+2t(C_{f_3})$. This proves the assertion.

§3. BOUNDS FOR $N(A,B)$

Since the curve $C = C_{A,B}$ has genus 13 if $\lambda = A^2+A+1+B \neq 0$ the Hasse-Weil-Serre bound for the number of $\mathbf{F}_q$-rational points $\#C_{A,B}(\mathbf{F}_q)$ says

$$q+1-13[2\sqrt{q}] \le \#C_{A,B}(\mathbf{F}_q) \le q+1+13[2\sqrt{q}]\,. \tag{12}$$

The number $N(A,B)$ of S_4-orbits of solutions of (1) with distinct $x_i \in \mathbf{F}_q$ satisfies

$$N(A,B) = (\#C_{A,B}(\mathbf{F}_q) - \text{contribution of } x=0,1,\infty)/24\,.$$

If $\mathrm{Tr}(A+1)=0$ we have 12 rational points in the fibres above $0,1,\infty$, while there are none if $\mathrm{Tr}(A+1)=1$. Then (12) implies for $N(A,B)$ the inequalities

$$(q-11-13[2\sqrt{q}])/24 \leq N(A,B) \leq (q+1+13[2\sqrt{q}])/24\,.$$

By employing the decomposition of the Jacobian, especially Corollary (2.4), and taking into account that the possible values of the trace of Frobenius t of supersingular elliptic curves are $t=0,\pm\sqrt{2q}$ for $q=2^m$ with m odd we can refine these bounds and we obtain our main result on the numbers $N(A,B)$:

(3.1) THEOREM. *For $q=2^m$ with m odd the number $N(A,B)$ satisfies the following inequalities:*

$$(q-11-2\sqrt{2q}-8[2\sqrt{q}])/24 \leq N(A,B) \leq (q+1+2\sqrt{2q}+8[2\sqrt{q}])/24\,. \tag{13}$$

Proof. The curve C_{f_1} is a supersingular elliptic curve, which implies that $-2\sqrt{2q} \leq 2t(C_{f_1}) \leq 2\sqrt{2q}$. Since the curve C_{f_3} has genus 1, $C_{f_1+f_3}$ has genus 2 and C_{g_λ} has genus 2 we obtain from the Hasse-Weil-Serre bound

$$-8[2\sqrt{q}] \leq 2t(C_{f_3})+2t(C_{f_1+f_3})+t(C_{g_\lambda}) \leq 8[2\sqrt{q}]\,.$$

Then it follows from Corollary (2.4) that the trace of Frobenius of $C_{A,B}$ is in the interval

$$[-2\sqrt{2q}-8[2\sqrt{q}],2\sqrt{2q}+8[2\sqrt{q}]]$$

which yields (13).

In the following table we illustrate this by listing the intervals in which the numbers lie according to (13).

TABLE 1

q	32	128	512	2048	8192
interval	$[0,4]$	$[0,14]$	$[4,38]$	$[50,120]$	$[270,412]$

For some further reflections on $N(A,B)$ we restrict our attention to the case $q=2^m$ with m odd. The practice of searching for curves with many points tells us that is is highly improbable that in a fibre product of curves the traces of Frobenius of the individual components simultaneously reach their maximal (or minimal) value. Hence it is very unlikely that the bounds given in (13) will be reached.

We intend to design an interval which contains almost all values of $N(A, B)$ using the description of C as a fibre product of the curves C_{f_i} for $i = 1, 2, 3$ given in (11) and a probabilistic argument on the distribution of traces of Frobenius.

The curves C_{f_1} and C_{f_2} are supersingular elliptic curves with the same trace of Frobenius $t = t(C_{f_i}) = 0, \pm\sqrt{2q}$. The curve $C_{f_1+f_2}$ has genus 3 which implies

$$-3[2\sqrt{q}] \le t(C_{f_1+f_2}) \le 3[2\sqrt{q}]\,.$$

So the trace of Frobenius for the normalization of the fibre product $C_{f_1} \times_{\mathbf{P}^1} C_{f_2}$ satisfies

$$-3[2\sqrt{q}] - 2\sqrt{2q} \le t \le 3[2\sqrt{q}] + 2\sqrt{2q}\,.$$

We compute bounds for the number of $x \in \mathbf{P}^1 - \{0, 1, \infty\}$ above which we have 4 points in the fibre of $C_{f_1} \times_{\mathbf{P}^1} C_{f_2}$ If $\mathrm{Tr}(A+1) = 0$ we find in total 8 points above $x = 0, 1, \infty$, while we find none if $\mathrm{Tr}(A+1) = 1$. Subsequently we take into account that completely splitting $x \in \mathbf{P}^1 - \{0, 1, \infty\}$ occur in pairs $(x, 1/x)$ and we obtain the following proposition.

(3.2) PROPOSITION. *If we let*

$$M(f_1,f_2) = \frac{1}{2}\#\{x \in \mathbf{P}^1(\mathbf{F}_q) - \{0, 1, \infty\} :\ x \text{ splits completely in } C_{f_1} \times_{\mathbf{P}^1} C_{f_2}\}$$

then we have for $\mathrm{Tr}(A+1) = 0$

$$\frac{q - 7 - 3[2\sqrt{q}] - 2\sqrt{2q}}{8} \le M(f_1,f_2) \le \frac{q - 7 + 3[2\sqrt{q}] + 2\sqrt{2q}}{8}$$

and for $\mathrm{Tr}(A+1) = 1$

$$\frac{q + 1 - 3[2\sqrt{q}] - 2\sqrt{2q}}{8} \le M(f_1,f_2) \le \frac{q + 1 + 3[2\sqrt{q}] + 2\sqrt{2q}}{8}\,.$$

We now consider the effect of the elliptic curve C_{f_3} in the fibre product. The j-invariant of C_{f_3} is $\lambda^{-4} \in \mathbf{F}_q^*$. This implies that $t(C_{f_3})$ is odd. For $\mathrm{Tr}(A+1) = 0$ we have $t(C_{f_3}) \equiv 1 \pmod 4$ and there are 4 rational points together above $x = 0, 1, \infty$, while if $\mathrm{Tr}(A+1) = 1$ we have $t(C_{f_3}) \equiv 3 \pmod 4$ and 2 rational points above $0, 1, \infty$. Furthermore, each element of $\mathbf{F}_q^*$ occurs exactly once as j-invariant in the family of curves C_{f_3}. That implies that $t(C_{f_3})$ assumes each odd integer value in the interval $[-[2\sqrt{q}], [2\sqrt{q}]]$. So the number of completely splitting pairs assumes each integral value in the intervals mentioned in the following proposition.

(3.3) PROPOSITION. *If we let*

$$M(f_3) = \frac{1}{2}\#\{x \in \mathbf{P}^1(\mathbf{F}_q) - \{0,1,\infty\} : \ x \text{ splits completely in } C_{f_3}\}$$

then $M(f_3)$ *assumes all integer values in the intervals*

$$\Big[\frac{q-3-[2\sqrt{q}]}{4}, \frac{q-3+[2\sqrt{q}]}{4}\Big] \qquad \text{if } \mathrm{Tr}(A+1)=0,$$

$$\Big[\frac{q-1-[2\sqrt{q}]}{4}, \frac{q-1+[2\sqrt{q}]}{4}\Big] \qquad \text{if } \mathrm{Tr}(A+1)=1.$$

Finally, we combine the two preceding propositions via a heuristic argument. Let

$$M(f_1,f_2,f_3) = \frac{1}{2}\#\{x \in \mathbf{P}^1(\mathbf{F}_q) - \{0,1,\infty\} : x \text{ splits completely on } C_{f_1} \times_{\mathbf{P}^1} C_{f_2} \times_{\mathbf{P}^1} C_{f_3}\}$$

Since there are $(q-2)/2$ pairs $(x, 1/x)$ $(x \neq 0,1,\infty)$ we expect

$$2\Big(\frac{q-3-[2\sqrt{q}]}{4}\Big)\Big(\frac{q-7-3[2\sqrt{q}]-2\sqrt{2q}}{8}\Big)/(q-2) \leq M(f_1,f_2,f_3)$$
$$\leq 2\Big(\frac{q-1+[2\sqrt{q}]}{4}\Big)\Big(\frac{q+1+3[2\sqrt{q}]+2\sqrt{2q}}{8}\Big)/(q-2).$$

If we work this out and neglect terms of order $1/\sqrt{q}$ and lower we find

$$\frac{q-4[2\sqrt{q}]-2\sqrt{2q}+4+4\sqrt{2}}{16} \leq M(f_1,f_2,f_3) \leq \frac{q+4[2\sqrt{q}]+2\sqrt{2q}+14+4\sqrt{2}}{16}. \tag{14}$$

Each completely splitting pair yields 16 solutions of (1) so to estimate the number of S_4-orbits of solutions $N(A,B)$ we multiply the interval by $16/24$ to get an interval I. Since $N(A,B)$ is even we adapt the endpoints of the interval I just obtained slightly. Namely we consider the smallest interval with endpoints the positive even integers which contains I and we denote this interval by I^{even}.

(3.4) HEURISTICS. *The odds are that the values of* $N(A,B)$ *are in the interval*

$$I^{\text{even}} = \Big[\frac{q-4[2\sqrt{q}]-2\sqrt{2q}+4\sqrt{2}+4}{24}, \frac{q+4[2\sqrt{q}]+2\sqrt{2q}+4\sqrt{2}+14}{24}\Big]^{\text{even}}.$$

We illustrate this by a little table.

TABLE 2

q	32	128	512	2048	8192
interval	$[0,6]$	$[0,12]$	$[10,34]$	$[64,108]$	$[300,384]$

§4. NUMERICAL RESULTS

In order to obtain numerical results on $N(A,B)$ to test our heuristics the first remark is that $N(A_1,B) = N(A_2,B)$ if $\mathrm{Tr}(A_1) = \mathrm{Tr}(A_2)$. So we have to distinguish only between $\mathrm{Tr}(A) = 0$ and $\mathrm{Tr}(A) = 1$. We shall compute the trace of Frobenius for the seven factors of our Jacobian. We shall write $f_4 = f_1 + f_2$, $f_5 = f_1 + f_3$, $f_6 = f_2 + f_3$ and $f_7 = f_1 + f_2 + f_3$. The Jacobians of the curves C_{f_i} given by $y^2 + y = f_i$ for $i = 1, \dots, 7$ constitute the seven factors of $\mathrm{Jac}(C_{A,B})$. We write

$$n_{f_i} = \#\{x \in \mathbf{F}_q^* : \mathrm{Tr}(f_i(x)) = 0\}.$$

(4.1) PROPOSITION. *The number of solutions $N(A,B)$ over $\mathbf{F}_{q=2^m}$ with m odd of the system (1) with $\lambda = A^2 + A + 1 + B \neq 0$ is given by*

$$N(A,B) = \frac{2q - 2 - 2(n_{f_1} + n_{f_2} + n_{f_3} - n_{f_4} - n_{f_5} - n_{f_6} + n_{f_7})}{24} \qquad \text{if } \mathrm{Tr}(A) = 0,$$

and

$$N(A,B) = \frac{-6q - 2 + 2\sum_{i=1}^{7} n_{f_i}}{24} \qquad \text{if } \mathrm{Tr}(A) = 1.$$

Proof. As just explained we may take $A = 0$ or $A = 1$. Then $\lambda = B + 1 \neq 0$ and we set $f_1 = (B+1)(x^3 + x)$, $f_2 = (B+1)(1/x^3 + 1/x)$ and $f_3 = (B+1)(x + 1/x)$. Then $C_{1,B} = C_{f_1} \times_{\mathbf{P}^1} C_{f_2} \times_{\mathbf{P}^1} C_{f_3}$ and $C_{0,B} = C_{f_1+1} \times_{\mathbf{P}^1} C_{f_2+1} \times_{\mathbf{P}^1} C_{f_3+1}$. As in Theorem (2.3) the curves C_{f_i} for $i = 4, \dots, 7$ give the remaining traces of Frobenius.

The trace of Frobenius $t(C_{f_i})$ is of the form

$$t(C_{f_i}) = q + 1 - 2n_{f_i} - a_i,$$

where a_i is the contribution of $x = 0, \infty$, while the trace of Frobenius of C_{f_i+1} is

$$t(C_{f_i}) = -q + 3 + 2n_{f_i} - b_i,$$

where b_i is the contribution of $x = 0, \infty$. By analyzing these contributions from 0 and ∞ one gets the proposition.

We now give tables with the distribution of the numbers $N(A,B)$ for $q=2^m$ with m odd and $5 \le m \le 13$. These tables are obtained by computing the numbers n_{f_i} and they solve the coset weight distribution problem for the corresponding $BCH(3)$ codes. The first unknown case up to now was $q=2^9$, see [C-Z]. Moreover, the tables confirm our heuristics. We list the frequencies divided by $q/2$.

TABLE 3

$q=2^5$:

$N(A,B)$	0	2
frequency	27	35

$q=2^7$:

$N(A,B)$	0	2	4	6	8	10
frequency	2	28	98	84	35	7

$q=2^9$:

$N(A,B)$	12	14	16	18	20	22	24	26	28	30	32
frequency	18	21	117	180	148	195	199	81	36	18	9

$q=2^{11}$:

$N(A,B)$	66	68	70	72	74	76	78	80	82	84	86
frequency	22	66	88	55	176	264	187	374	374	374	451
$N(A,B)$	88	90	92	94	96	98	100	102	104	106	108
frequency	365	341	275	341	154	44	55	33	11	22	22

$q=2^{13}$:

In this case we encounter a new phenomenon. The function $N(A,B)$ assumes even values in the interval $[290,390]$, but not all even values are taken. This contradicts the expectation of [C-Z] that the values form a sequence of even integers without gaps. The frequency divided by $q/2$ of the value $290+2\ell$ with $0 \le \ell \le 50$ is given by

$$13\,\gamma_\ell + \begin{cases} 1 & \text{if } \ell = 11, \\ 1 & \text{if } \ell = 37, \\ 0 & \text{else,} \end{cases}$$

where $\gamma = (\gamma_0, \ldots, \gamma_{50})$ is the vector

$$\gamma = (1,0,1,0,1,0,6,3,5,5,12,7,19,15,22,25,37,40,43,37,35,60,54,72,72,\\ 58,65,61,57,57,63,48,35,44,34,34,25,29,25,15,9,7,2,3,7,3,3,1,0,1,2).$$

In accordance with our heuristics less than 1% of the $N(A,B)$ lie outside the interval $[300,384]$.

§5. The covering radius

A problem in coding theory that precedes the coset weight distribution problem is the determination of the covering radius. It is defined for a binary linear code C of length n as the smallest integer ρ such that the spheres of radius ρ around the codewords cover $\mathbf{F}_2^n$. Equivalently, it is the maximum weight of a coset leader (by which we mean a vector of minimum weight in a coset of C in $\mathbf{F}_2^n$). It is an interesting parameter of a code since it provides information on the performance of the code when used in data compression.

In a series of papers [H-B], [A-M] and [H], of which [H-B] and [H] treat the case m even and [A-M] the case m odd, it was proved that the $BCH(3)$ code of length $n = 2^m - 1$ has covering radius

$$\rho(BCH(3)) = 5 \qquad \text{for} \quad m \ge 4\,.$$

The proofs for the various cases are very different. Using algebraic geometry we can give a unified proof.

In order to prove that $\rho(BCH(3)) = 5$ we have to show that for every $(A,B,C) \in \mathbf{F}_q^3$ the system of equations:

$$\begin{aligned} x_1 + \ldots + x_5 &= A\,, \\ x_1^3 + \ldots + x_5^3 &= B\,, \\ x_1^5 + \ldots + x_5^5 &= C\,, \end{aligned} \tag{15}$$

has a solution $(x_1, \ldots, x_5) \in \mathbf{F}_q^5$. On replacing x_i by $x_i + A$ we may assume without loss of generality that $A = 0$ and $(B,C) \neq (0,0)$. If we then homogenize (15) the system

$$\sum_{i=1}^{5} x_i = 0, \qquad \sum_{i=1}^{5} x_i^3 = Bx_0^3, \qquad \sum_{i=1}^{5} x_i^5 = Cx_0^5 \tag{16}$$

defines a projective variety V of dimension 2 in the five dimensional projective space $\mathbf{P}^5$.

We intersect V with the hyperplane $x_0 + x_5 = 0$ and obtain a system of equations of the form (2). By using the results of Section 1 (especially Corollary (1.3)) one can easily show that $\rho(BCH(3)) = 5$ for $m \geq 10$. We leave the details to the reader.

As a final remark we would like to point out that we think that many more problems on cyclic codes can be attacked succesfully using methods from algebraic geometry as is done in this paper. We refer to [C] for a list of such problems.

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