3. P_C^2 AND THE 4-SPHERE S^4

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that we considered in Sections 2 and 3 above. Similarly, if G is $\mathbf{Z}/2\mathbf{Z}$ acting on Q as the antipodal map, then the corresponding extension to $P^2_{\mathbf{C}}$ is given by complex conjugation.

3. $P^2_{\mathbf{C}}$ AND THE 4-SPHERE S^4

The previous discussion, restricted to $n = 2$ and compared to the cohomogeneity 1 isometric action of $\mathrm{SO}(3,\mathbf{R})$ on S^4 constructed in [HL], motivates an equivariant version of the Arnold-Kuiper-Massey theorem [Ar1, Ar2, Ku, Ma1], saying that $P^2_{\mathbf{C}}$ modulo conjugation is the 4-sphere. In this section we give a new proof of this theorem. We construct an explicit algebraic map $\Phi\colon P^2_{\mathbf{C}} \to S^4$, which is equivariant with respect to the cohomogeneity 1 isometric actions of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$ and S^4 and induces a diffeomorphism $P^2_{\mathbf{C}}/\mathit{conjugation} \cong S^4$.

We start by recalling the $\mathrm{SO}(3,\mathbf{R})$-action on S^4, as explained by Hsiang and Lawson in [HL; Example 1.4].

Let $\mathcal{S}$ be the vector space of real 3×3, traceless and symmetric matrices. As a real vector space $\mathcal{S}$ is $\mathbf{R}^5$, and it can be equipped with a metric given by the inner product $(A,B) \mapsto \mathrm{trace}(AB)$. Let $\mathcal{S}^{(4)}$ be the space of matrices in $\mathcal{S}$ with norm 1. One has an obvious diffeomorphism $S^4 \cong \mathcal{S}^{(4)}$, which becomes isometric if we endow S^4 with its usual round metric and $\mathcal{S}^{(4)}$ with the metric given by the inner product in $\mathcal{S}$. We shall identify these two spaces in the sequel, denoting both of them by S^4 indistinctly. The group $\mathrm{SO}(3,\mathbf{R})$ acts on $\mathcal{S}$ by $A \mapsto O^tAO$, where O^t is the transposed matrix (which is equal, in our case, to O^{-1}). This induces an isometric action Γ of $\mathrm{SO}(3,\mathbf{R})$ on S^4. This action on S^4 has two disjoint copies of $P^2_{\mathbf{R}}$ as special fibres (see the remark at the end of this section). The space of orbits is the interval $[0,1]$, with the endpoints giving the special orbits. Each principal orbit (i.e. the orbits of highest dimension) is a flag manifold

$$F^3(2,1) \cong \mathrm{SO}(3,\mathbf{R})\,/\,(\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}) \cong L(4,1)\,/\,(\mathbf{Z}/2\mathbf{Z}),$$

of pairs (P,l) with P a plane in $\mathbf{R}^3$ and l line in P, where $L(4,1)$ is the lens space $S^3\,/\,(\mathbf{Z}/4\mathbf{Z}) \cong \mathrm{SO}(3,\mathbf{R})\,/\,(\mathbf{Z}/2\mathbf{Z})$.

Let us give a similar description of $P^2_{\mathbf{C}}$. Let

$$\mathfrak{H}(3,\mathbf{C}) = \{H \in M(3,\mathbf{C}) \mid H = H^*\}$$

be the space of complex 3×3 Hermitian matrices, where $H^* = \overline{H}^t$ is the adjoint matrix of H, obtained by first conjugating each entry of H and then transposing the matrix. We equip $\mathfrak{H}(3, \mathbf{C})$ with the Hermitian inner product

$$\langle H_1, H_2 \rangle = \frac{1}{2} \operatorname{trace}(H_1 H_2) . \tag{3.1}$$

As a vector space, with this inner product, $\mathfrak{H}(3, \mathbf{C})$ is the ordinary Euclidean space $\mathbf{E}^9$. Consider the subset $P(2)$ of $\mathfrak{H}(3, \mathbf{C})$ defined by

$$P(2) = \left\{ H \in \mathfrak{H}(3, \mathbf{C}) \mid H^2 = H \quad \text{and} \quad \operatorname{trace}(H) = 1 \right\} . \tag{3.2}$$

LEMMA 3.3. *The set $P(2)$ is a manifold, diffeomorphic to $P^2_{\mathbf{C}}$. Moreover, if we endow $P(2)$ with the metric defined by (3.1), then $P(2)$ is isometric to $P^2_{\mathbf{C}}$ equipped with the Fubini-Study metric (of constant holomorphic sectional curvature 4).*

We remark that it is possible to describe $P^n_{\mathbf{C}}$ in a similar way, but we restrict our attention to $n = 2$ because this is all we need.

Proof. We claim that if H is in $P(2)$, then it is an orthogonal projection over a complex line. In fact, if H is in $P(2)$, then it is diagonalizable by a unitary matrix and its eigenvalues are 0 or 1, because $H^2 = H$. Since the trace is one, two eigenvalues must be 0 and the other is 1. Hence H is a surjection of $\mathbf{C}^3$ over a complex line, and this map has to be an orthogonal projection because H is Hermitian. Conversely, it is clear that each line $L \in \mathbf{C}^3$ determines a unique orthogonal projection of $\mathbf{C}^3$, and this is given by a matrix in $P(2)$. The diffeomorphism in Lemma 3.3 is achieved by the map that carries H into the corresponding line in $\mathbf{C}^3$. To prove that this map gives a metric equivalence, we notice that the unitary group $U(3)$ acts on $\mathfrak{H}(3, \mathbf{C})$ by $H \mapsto U^* H U$, and $P(2)$ is an orbit of this action, with isotropy $\big(U(2) \times U(1)\big)$. Thus,

$$P(2) \cong U(3) / \big(U(2) \times U(1)\big) \cong P^2_{\mathbf{C}} ,$$

and the metric on $P(2)$ is obviously $U(3)$-invariant. Hence the induced metric on $P^2_{\mathbf{C}}$ is also $U(3)$-invariant, and this characterizes the Fubini-Study metric, up to scaling. $\square$

We recall now that the quotient of $P^2_{\mathbf{C}}$ by the complex conjugation j is a smooth manifold, which is not an obvious fact since j has fixed points. This is carefully explained in [Mar], so we only sketch a few ideas here. Away from the fixed point set $\Pi \cong P^2_{\mathbf{R}}$, the involution j is free, so the quotient is a smooth manifold. The problem is on Π. A tubular neighbourhood of

Π in $P^2_{\mathbf{C}}$ can be regarded as an open disk normal bundle, and conjugation carries each normal fibre into itself. Since the quotient of each normal 2-disk by the involution is again a 2-disk, it follows that the quotient $P^2_{\mathbf{C}}/j$ is a topological manifold. Making this argument more carefully one gets that $P^2_{\mathbf{C}}/j$ is in fact a PL-manifold, as noticed in [Ku], and therefore it is smooth, since every piecewise linear 4-manifold is smooth. In [Mar] Marin defines the smooth structure on $P^2_{\mathbf{C}}/j$ directly, without using PL-structures. An important point is that the smooth structure on $P^2_{\mathbf{C}}/j$ is such that the obvious projection $P^2_{\mathbf{C}} \to P^2_{\mathbf{C}}/j$ is differentiable.

Let us denote by Γ the aforementioned isometric action of $\mathrm{SO}(3,\mathbf{R})$ on S^4, and by $\widetilde{\Gamma}$ the standard action of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$, which is by isometries with respect to the Fubini-Study metric. This action is defined either by considering $\mathrm{SO}(3,\mathbf{R})$ as a subgroup of $O(3,\mathbf{C})$, acting on the space of lines in $\mathbf{C}^3$, or via the action of $\mathrm{SO}(3,\mathbf{R})$ on the space of matrices $P(2) \subset H(3,\mathbf{C})$ given by

$$(O,A) \mapsto O^tAO\,.$$

By Lemma 3.3, both metrics on $P^2_{\mathbf{C}}$ are equivalent; also for every $O \in \mathrm{SO}(3,\mathbf{R})$, $H \in P(2)$ and $v \in \mathbf{C}^3$ such that $H(v) = v$, one has $O^tHO(O^{-1}(v)) = O^{-1}(v)$, because $O^{-1} = O^t$. Hence both actions on $P^2_{\mathbf{C}} \cong P(2)$ are equivalent. Similarly, given the $\mathrm{SO}(3,\mathbf{R})$-actions $\widetilde{\Gamma}$ on $P^2_{\mathbf{C}}$ and Γ on S^4, we say that these actions are equivariant if there exists a map $\Phi\colon P^2_{\mathbf{C}} \to S^4$ which makes the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{SO}(3,\mathbf{R}) \times P^2_{\mathbf{C}} & \xrightarrow{\widetilde{\Gamma}} & P^2_{\mathbf{C}} \\
{\scriptstyle Id\times\Phi}\big\downarrow & & {\scriptstyle\Phi}\big\downarrow \\
\mathrm{SO}(3,\mathbf{R}) \times S^4 & \xrightarrow{\Gamma} & S^4\,.
\end{array}$$

In this case we say that Φ *conjugates* the actions Γ and $\widetilde{\Gamma}$. The map Φ carries orbits into orbits, i.e. the decompositions of $P^2_{\mathbf{C}}$ and S^4 into orbits are (smoothly) equivalent.

Let us now state the equivariant Arnold-Kuiper-Massey theorem:

THEOREM 3.4. *There is a real algebraic equivariant map $\Phi\colon P^2_{\mathbf{C}} \to S^4$, which is invariant by the complex conjugation j and induces a diffeomorphism $P^2_{\mathbf{C}}/j \cong S^4$, providing a conjugation between the isometric* $\mathrm{SO}(3,\mathbf{R})$*-actions $\widetilde{\Gamma}$ on $P^2_{\mathbf{C}}$ and Γ on S^4.*

We notice that Theorem 3.4, together with [HL], imply that the image of $P^2_{\mathbf{R}} \subset P^2_{\mathbf{C}}$ under the above map is the image of $P^2_{\mathbf{R}}$ by the classical Veronese embedding $(P^2_{\mathbf{C}}, P^2_{\mathbf{R}}) \hookrightarrow (P^5_{\mathbf{C}}, S^4)$.

The proof of Theorem 3.4 follows from several lemmas below.

LEMMA 3.5. *Let A be a real (3×3)-matrix. Then A is the real part of a matrix H in $P(2)$ if and only if*

i) *A is symmetric with trace 1;*

ii) *A has 0 as an eigenvalue and the other two eigenvalues λ_i and λ_j are roots of an equation of the form:*

$$\lambda^2 - \lambda + k = 0 ,$$

for some constant $k \in \mathbf{R}$ with $0 \leq k \leq \frac{1}{4}$.

If A and H are as above, and if $O \in \mathrm{SO}(3, \mathbf{R})$ is such that $O^t A O$ is a diagonal matrix, then the imaginary part B of H, taken into its canonical form $O^t B O$, has only two possible non-zero entries, which are $\pm\sqrt{k}$. In particular, if $k = 0$, then $H = A$.

Proof. Let us consider a matrix $H \in P(2)$ and decompose it into its real and imaginary parts: $H = A + iB$. Then one has $\overline{H}^t = A^t - iB^t$. Also $H = \overline{H}^t$ because H is Hermitian. Hence $A = A^t$ and $B = -B^t$, i.e. A is symmetric and B is anti-symmetric. Thus the trace of A is 1, proving statement (i). One also has

$$H^2 = A^2 - B^2 + i(AB + BA) ,$$

and $H^2 = H$ because H is in $P(2)$. Therefore $A = A^2 - B^2$ and $B = AB + BA$.

Now, A is symmetric, and so is A^2; these two matrices obviously commute, so they can be diagonalized simultaneously by a matrix $O \in \mathrm{SO}(3, \mathbf{R})$. Since $B^2 = A^2 - A$, one knows that $O^t B^2 O$ is also diagonal:

$$O^t B^2 O = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} ,$$

with $\mu_i = \lambda_i^2 - \lambda_i$, for each $i = 1, 2, 3$, where the λ_i are the eigenvalues of A. But B is antisymmetric and commutes with B^2, which is symmetric. Hence the same matrix O takes B to its canonical form:

$$O^t B O = \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix}$$

for some $a, b, c \in \mathbf{C}$. This implies that

$$O^tB^2O = (O^tBO)(O^tBO) = \begin{pmatrix} -a^2-c^2 & -bc & ab \\ -bc & -a^2-b^2 & -ac \\ ab & -ac & -b^2-c^2 \end{pmatrix},$$

which we know is a diagonal matrix. Therefore two of the numbers a, b, c must be zero. Assume for instance that a and b are 0, then both eigenvalues λ_1 and λ_3 are roots of the polynomial

$$\lambda^2 - \lambda + c^2 = 0\,.$$

This implies that

$$\lambda_1 + \lambda_3 = 1 \qquad \text{and} \qquad \lambda_1 \cdot \lambda_3 = c^2 \geq 0\,.$$

Hence $\lambda_2 = 0$ (because the trace of A is 1), so 0 is an eigenvalue of A. The other eigenvalues λ_1 and λ_3 must both be ≥ 0 and ≤ 1, because their product is non-negative and their sum is 1. Moreover the roots must be real, therefore $k = c^2 \leq \frac{1}{4}$, proving statement (ii).

Also, in this case the eigenvalues of A determine the imaginary part B of H up to sign:

$$B = \pm O \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} O^t\,,$$

with $c^2 = \lambda_1 - \lambda_3^2 = \lambda_3 - \lambda_3^2$, proving in this case the last statement of Lemma 3.5. The other cases, when either $a = c = 0$ or $b = c = 0$, are similar to the previous one. This proves that if $A = \Re(H)$ for some matrix $H \in P(2)$, then A is as stated in Lemma 3.5. Conversely, given A satisfying these conditions, the above arguments tell us how to construct B so that these matrices are the real and imaginary parts of some H in $P(2)$. $\square$

Now, given $H \in P(2)$, its real part is $\Re(H) = \frac{1}{2}(H + \overline{H})$. Define

$$\psi\colon P(2) \to M(3, \mathbf{R})\,,$$

the space $M(3, \mathbf{R})$ being the space of real (3×3)-matrices, by the formula

$$\psi(H) = \frac{1}{3} I_3 - \Re(H) \in M(3, \mathbf{R})\,, \tag{3.6}$$

where I_3 is the (3×3)-identity matrix. In other words, $\psi(H)$ is the real part of the matrix $(\frac{1}{3}I_3 - H)$. Since $H \in P(2)$, it follows that $\psi(H)$ is actually contained in $\mathcal{S}$.

It is clear that the above action of $\mathrm{SO}(3, \mathbf{R})$ on $P(2)$ given by conjugation is equivalent, via the above diffeomorphism $P(2) \cong P^2_{\mathbf{C}}$, with the standard action studied in §2 and §3 above. It is also clear that, for every $O \in \mathrm{SO}(3, \mathbf{R})$, one has

$$\psi(O^tHO) = \frac{1}{3}I - \frac{1}{2}\big(O^t(H+\overline{H})O\big) = O^t\big(\frac{1}{3}I - \frac{1}{2}(H+\overline{H})\big)O = O^t\psi(H)O.$$

Hence we have

LEMMA 3.7. *The map ψ is equivariant. That is, for every $O \in \mathrm{SO}(3, \mathbf{R})$ and $H \in P(2)$, one has $\psi(O^tHO) = O^t\psi(H)O$.*

LEMMA 3.8. *Given $S \in \mathcal{S} - \{0\}$, there exists a unique positive $t \in \mathbf{R}$, such that the matrix $(\frac{1}{3}I - tS)$ is the real part of some matrix $H \in P(2)$.*

Proof. By Lemma 3.7, we may assume that S is diagonal. Hence the matrix $\widehat{S}_t = (\frac{1}{3}I - tS)$ is also diagonal, say

$$\widehat{S}_t = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{pmatrix}$$

with $\lambda_i(t) = \frac{1}{3} - t\mu_i$, where the μ_i are the eigenvalues of S. We notice that for all $t \in \mathbf{R}$, one has

$$\operatorname{trace} \widehat{S}_t = 1 - t(\operatorname{trace} S) = 1,$$

because S has trace 0. Hence all these matrices satisfy condition (i) of Lemma 3.5.

Let us look for the possible values of t that give solutions of Lemma 3.5. That is, we want $t > 0$ for which one eigenvalue $\lambda_i(t)$ is 0 and the others are such that their sum is 1 and their product is ≥ 0 and $\leq \frac{1}{4}$.

Let us number the eigenvalues of S so that $\mu_1 \leq \mu_2 \leq \mu_3$. Since their sum is 0 and S is not the zero matrix, one must have $\mu_1 < 0$ and $\mu_3 > 0$. If we want t as above, one $\lambda_i(t)$ must vanish. Let us look for solutions with $\lambda_1(t) = 0$. This means that $t = \frac{1}{3\mu_1} < 0$, and we want $t > 0$. Hence, there are no solutions with $\lambda_1(t) = 0$.

Now let us look for solutions with $\lambda_2(t) = 0$. This implies that $t = \frac{1}{3\mu_2}$; for this to be possible we must have $\mu_2 \neq 0$. If $\mu_2 < 0$, then $t < 0$ and we want t to be positive. Thus, we only care about $\mu_2 > 0$. We have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_2}) \quad \text{and} \quad \lambda_3(t) = \frac{1}{3}(1 - \frac{\mu_3}{\mu_2}).$$

We have $\mu_1 < 0 < \mu_2$, so $\lambda_1(t) > 0$. If $\mu_2 < \mu_3$, then $\lambda_3(t) < 0$, thus the product $\lambda_1(t)\lambda_3(t)$ is < 0, so there are no such solutions to Lemma 3.8. The other possibility is $\mu_2 = \mu_3$; this also implies $\lambda_3(t) = 0$. In this case one has $\lambda_1(t) = 1$ and $\lambda_2(t) = \lambda_3(t) = 0$, and $t = \frac{1}{3\mu_2}$ is positive. Hence we have a solution, and this is unique because $\mu_2 = \mu_3$. If $\mu_2 = 0$, then $\lambda_2(t)$ cannot be 0 and we cannot find solutions like this.

Summarizing, so far we have seen that: i) there are no solutions as in Lemma 3.8 for which $\lambda_1(t) = 0$; ii) if $\mu_2 \leq 0$, there are no solutions as in Lemma 3.8 for which $\lambda_2(t) = 0$; and iii) if $\mu_2 = \mu_3$, then there is a unique solution as in Lemma 3.8, for which $\lambda_2(t) = \lambda_3(t) = 0$ and $\lambda_1(t) = 1$.

Finally, let us look for solutions with $\lambda_3(t) = 0$, i.e. with $t = \frac{1}{3\mu_3}$. We know, by hypothesis, that $\mu_2 \leq \mu_3$ and $\mu_3 > 0$. If $\mu_2 = \mu_3$, then we are in the previous case and there is a unique positive t giving a solution as in Lemma 3.8. Let us assume now that $\mu_2 < \mu_3$. Then we have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_3}) \quad \text{and} \quad \lambda_2(t) = \frac{1}{3}(1 - \frac{\mu_2}{\mu_3}),$$

which are both ≥ 0. Since their sum is 1, it follows that each $\lambda_i(t)$ is also ≤ 1.

The product of $\lambda_1(t)$ and $\lambda_2(t)$ satisfies

$$\begin{aligned} 0 \leq \lambda_1(t) \cdot \lambda_2(t) &= \frac{1}{9}(1 - \frac{\mu_1 + \mu_2}{\mu_3} + \frac{\mu_1\mu_2}{\mu_3^2}) = \frac{1}{9}(2 + \frac{\mu_1\mu_2}{\mu_3^2}) \\ &= \frac{1}{9}(2 + \frac{\mu_1\mu_2}{(\mu_1 + \mu_2)^2}) \leq \frac{1}{4}, \end{aligned}$$

since $\mu_1 + \mu_2 + \mu_3 = 0$ and $\frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2} \leq \frac{1}{4}$ because $\frac{1}{4}(a+b)^2 \geq ab$ for any real numbers a and b (with equality if and only if $a = b$). Hence $t = \frac{1}{3\mu_3}$ is the unique solution satisfying the conditions of Lemma 3.8. □

We now "normalize" the map ψ so that its image is contained in $S^4 \subset \mathcal{S}$. For this we define a function

$$\alpha(H) = [\operatorname{trace}(\psi(H)^2)]^{-\frac{1}{2}},$$

i.e. $\alpha(H)$ is the inverse of the norm of $\psi(H)$ in $\mathcal{S}$, and we set

$$\Phi(H) = \alpha(H)\,\psi(H).$$

One has

$$\begin{aligned} \operatorname{trace}[\psi(H)^2] &= \operatorname{trace}[(\frac{1}{3}I_3 - \frac{1}{2}(H + \overline{H}))^2] \\ &= \operatorname{trace}[\frac{1}{9}I_3 - \frac{1}{3}(H + \overline{H}) + \frac{1}{4}(H^2 + \overline{H}^2 + H\overline{H} + \overline{H}H)] \\ &= \frac{1}{6} + \frac{1}{4}\operatorname{trace}(H\overline{H} + \overline{H}H), \end{aligned}$$

which is always positive since the matrix $(H\overline{H}+\overline{H}H)$ is positive semi-definite, so its trace is ≥ 0. Hence the maps α and Φ are well defined. It is clear that the image of Φ is contained in $S^4 \subset \mathcal{S}$, because the linearity of the trace implies that

$$[\mathrm{trace}(\Phi(H))]^2 = \alpha^2(H)\,[\mathrm{trace}\,\psi(H)]^2 = 1\,.$$

It is also clear that Φ is $\mathrm{SO}(3,\mathbf{R})$-equivariant, since the trace is invariant under conjugation and ψ is equivariant by Lemma 3.7. These considerations imply both Lemma 3.8 and the following

LEMMA 3.9. *The map Φ is an equivariant surjection from $P(2)$ over $S^4 \subset \mathcal{S}$, and it is two-to-one, except over the image of the real matrices in $P(2)$ where it is one-to-one.*

This gives the map in Theorem 3.4 that determines an equivariant diffeomorphism between S^4 and $P^2_{\mathbf{C}}$ modulo the involution given by conjugation. To complete the proof of Theorem 3.4 we need to show that Φ is invariant under the involution of $P(2)$ that corresponds to complex conjugation in $P^2_{\mathbf{C}}$. For this we notice that if L_H is the complex line in $\mathbf{C}^3$ which is the image of $H \in P(2)$, and if $0 \neq (z_1, z_2, z_3) \in L_H$, we can associate to H the point in $P^2_{\mathbf{C}}$ with projective coordinates $[z_1, z_2, z_3]$. To the matrix $\overline{H}$ there corresponds the line with projective coordinates $[\overline{z}_1, \overline{z}_2, \overline{z}_3]$. Therefore we have

LEMMA 3.10. *The involution $j*$ of $P(2)$ defined by $j*(H) = \overline{H}$ coincides with the involution j of $P^2_{\mathbf{C}}$ given by complex conjugation,* $[z_1, z_2, z_3] \overset{j}{\mapsto} [\overline{z}_1, \overline{z}_2, \overline{z}_3]$.

Then Φ is invariant under this involution, since $\Re(H) = \Re(\overline{H})$, proving Theorem 3.4. $\square$

4. SOME APPLICATIONS AND REMARKS

It is interesting to describe explicitly the orbits of the Γ action of $\mathrm{SO}(3,\mathbf{R})$ on S^4, regarded[2]) as the set of matrices with norm 1 in $\mathcal{S}$. In fact, the orbits of this action are conjugacy classes (or *congruency classes*) of traceless symmetric matrices whose square has trace 1. This is the connection between our construction and the spherical Tits buildings. Every $S \in \mathcal{S}$ can

[2]) This orbit description of S^4 is also given in [Ma2].