§3. KÄHLER MANIFOLDS

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We view a Kähler manifold as a Riemannian manifold W with a closed 2-form ω. Every submanifold V of dimension $n = 2m$ satisfies the Wirtinger inequality

$$\text{(3.0)} \qquad \operatorname{Vol}(V) \geq \int_V (\omega)^m ,$$

and equality

$$\text{(3.1)} \qquad \operatorname{Vol}(V) = \int_V (\omega)^m$$

holds if and only if V is complex analytic (of complex dimension m). Observe that (3.0) and (3.1) imply the Federer theorem.

Start now with a Kähler manifold X of real dimension $n = 2m$ and apply (3.1) to the iterated graph $(\Gamma_f)_k \subset X^k$ of an endomorphism $f \colon X \to X$. We get

$$\text{(3.2)} \qquad \operatorname{Vol}(\Gamma_f)_k = \left\langle \Big(\sum_\alpha^k\Big)^m, [X] \right\rangle ,$$

where $\alpha \in H^2(X, \mathbf{R})$ is the cohomology class represented by the structural 2-form, $\big(\sum_\alpha^k\big) = \sum_{i=1}^k (f^*)^i(\alpha)$, and $[X]$ is the fundamental class of X.

When $X = \mathbf{C}P^m$ and $\deg f = d = p^m$ we have

$$\Big(\sum_\alpha^k\Big) = \frac{p^{k+1} - 1}{p - 1}\, \alpha$$

and conclude that

$$\text{(3.3)} \qquad \operatorname{lov} \Gamma_f = \log \deg f .$$

Together with (1.0) this implies our main inequality (0.1).

REMARKS. (3.3) holds whenever α is an eigenvector of the operator $f^* : H^2(X, \mathbf{R}) \to H^2(X, \mathbf{R})$ but not generally, as shown by linear endomorphisms of tori.

When X is complex but not Kähler, neither "lov" nor entropy can be estimated in homological terms. Moreover, the entropy of a holomorphic vector field can be non-zero. (In the Kähler case, maps homotopic to the identity have "lov" $= 0$.)

Take a complex semi-simple Lie group and factor it by a discrete uniform subgroup. The group translations in this factor can have non-zero entropy. To be specific, we take $SL_2(\mathbf{C})$ acting by isometries in three-dimensional hyperbolic space. Thus geodesic flows on compact 3-dimensional hyperbolic manifolds are factors of translations of the above type and their entropy must be positive.

HOPF MANIFOLDS

The Hopf manifold H^m is diffeomorphic to $S^1 \times S^{2m-1}$. As a complex manifold it is the factor of $\mathbf{C}^m \setminus 0$ by the following action of $\mathbf{Z}$:

$$x \longmapsto z_0^r x\,, \qquad x \in \mathbf{C}^m \setminus 0\,,\ z_0 \in \mathbf{C}\,,\ |z_0| \neq 0, 1\,,\ r \in \mathbf{Z}\,.$$

There is a natural fibration $H^m \to \mathbf{C}P^{m-1}$ and each endomorphism of $\mathbf{C}P^{m-1}$ extends to H^m. When $m > 1$ Hopf manifolds are not Kähler; nevertheless, for any endomorphism $f\colon H^m \to H^m$ we have

$$h(f) = \operatorname{lov} f = \log \deg f\,. \tag{3.4}$$

Proof. Each endomorphism f preserves the fibers of the fibration $H^m \to \mathbf{C}P^{m-1}$ and "lov" is additive in the following sense.

Given a holomorphic fibration $H \to V$ with fibers T_v, $v \in V$, equiped with Kähler structures. Suppose that structural cohomology classes $\alpha_v \in H^2(T_v;\mathbf{R})$ are parallel under the holonomy action of $\pi_1(V)$. If $f\colon H \to H$ is a fiber-preserving endomorphism, one can define

$$\Big(\overset{k}{\textstyle\sum}_{\alpha}\Big) = \sum_{i=1}^{k} (f^*)^i(\alpha_{v_0}) \in H^2(T_{v_0};\mathbf{R})\,,$$

$v_0 \in V$, and an endomorphism $g\colon V \to V$ induced by f.

THE ADDITION FORMULA

$$\operatorname{lov} f = \operatorname{lov}\, g + \lim_{k\to\infty} \frac{1}{k} \log\Big\langle \Big(\overset{k}{\textstyle\sum}_{\alpha}\Big)^m, [T_{v_0}]\Big\rangle\,,$$

where $m = \dim_{\mathbf{C}} T_{v_0}$, and $[T_{v_0}]$ is the fundamental class of T_{v_0}.

This formula is almost as obvious as (3.2) and, together with (3.3), it yields (3.4).

GENERALIZATIONS AND PROBLEMS

In the previous discussion, we avoided mentioning the fact of the scarcity of complex endomorphisms. I am not able to add any interesting examples to those considered above [1]). Generic manifolds have no endomorphisms. Every surjective endomorphism f is finite-to-one and when $\deg f = 1$ it is injective. More generally, if V and V' are complex (not necessarily compact or Kähler) manifolds of equal dimensions and their even Betti numbers are finite and equal (i.e. $b_{2i} = b'_{2i}$) then every proper surjective holomorphic map $f\colon V \to V'$ is finite-to-one and when $\deg f = 1$ the map is injective. The finitness condition cannot be omitted; take $\mathbf{C}^2$, blow it up at all points from a lattice. The endomorphism of the resulting manifold induced by the transformation $\mathbf{C}^2 \to \mathbf{C}^2$, $x \longmapsto \frac{x}{2}$, has infinitely many blow-downs.

The lack of endomorphisms can be offset by abundance of general holomorphic graphs. The most regular asymptotic behavior is displayed by graphs $\Gamma \subset X \times X$ of finite type when both projections $\Gamma \to X$ are finite-to-one. In the finite type case the infinitely iterated graph Γ_∞ can be viewed as a $2m$-dimensional ($m = \dim_{\mathbf{C}} \Gamma = \dim_{\mathbf{C}} X$) compact set 'foliated' by complex m-dimensional leaves and having Cantor sets as transversal sections. The holomorphic finite type graphs probably have finite "lov" and entropy and at least in the Kähler case this can be proved as follows: Denote by $\gamma \in H^n(X \times X; \mathbf{R})$, $n = 2m$, $m = \dim_{\mathbf{C}} \Gamma = \dim_{\mathbf{C}} X$, the class dual to $[\Gamma]$ and by $\gamma_{i,i+1} \in H^n(X^k; \mathbf{R})$ the class induced from γ by projecting X^k onto the product of its i-th and $(i+1)$-th factors. Denote by $\beta \in H^2(X^k; \mathbf{R})$ the class represented by the structural 2-form. One can easily see that

$$\operatorname{Vol} \Gamma_k = \left\langle \beta^m \prod_{i=1}^{k-1} \gamma_{i,i+1}, [X^k] \right\rangle ;$$

thus "lov" is finite.

In the end, I must admit my inability to prove (or disprove) the inequality $h(V) \geq \operatorname{lov} \Gamma$ even when Γ is a graph of an endomorphism. (Of course, I mean here only the holomorphic case. For smooth endomorphisms, the situation $h(f) = 0$, $\operatorname{lov} f > 0$ occurs already for maps $\mathbf{S}^1 \to \mathbf{S}^1$ and, probably, for higher dimensions the opposite: $h(f) > 0$, $\operatorname{lov} f = 0$ can also happen.) The inequality $h(\Gamma) \geq \operatorname{lov} \Gamma$ reminds one of the Shub entropy conjecture [5] proposing a lower estimate for the entropy in homological terms. In the complex-analytic context, one has more homology theories to provide further speculations.

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[1]) There are some in the Appendix.