Appendix : Examples of holomorphic endomorphisms

Objekttyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

APPENDIX: EXAMPLES OF HOLOMORPHIC ENDOMORPHISMS

The following examples appeared after the discussion I had with Spencer Bloch and David Mumford.

(1) TWISTED HOPF MANIFOLDS

Divide $\mathbf{C}^n \setminus 0$ by the action of a linear operator A without eigenvalues in the unit disk. All endomorphisms of the quotient ($n > 1$) come from polynomial maps $\widetilde{f}\colon \mathbf{C}^n \to \mathbf{C}^n$. For such an endomorphism, its entropy and "lov" are probably equal to "log deg".

EXAMPLE. $A\colon (z_1, z_2) \longmapsto (\lambda_1 z_1, \lambda_2 z_2)$ and $\widetilde{f}\colon (z_1, z_2) \longmapsto (z_1^p, z_2^p)$.

(2) GENERALIZED HOPF MANIFOLDS

Let $f_0\colon X_0 \to X_0$ be an endomorphism. Take a line bundle L over X_0 such that $f_0^*(L) = \bigotimes^p L$ $(:= L \otimes \cdots \otimes L,\ \ p$ times$)$. Locating such an L is usually quite easy by looking at $\mathrm{Pic}(X_0)$. Denote by Y the total space of L. There is a fiberwise map $\widetilde{f}\colon Y \to Y$ lifting f_0 and acting on fibers as $z \longmapsto z^p$. If we divide Y by a fiberwise action of $\mathbf{Z}$ (it is $z \longmapsto z_0 z$, $z_0 \neq 0$, in each fiber) we get $f\colon Y/\mathbf{Z} \to Y/\mathbf{Z}$.

There is another way to compactify Y by taking the total space of the one-dimensional projective bundle associated to L. The endomorphism $\widetilde{f}$ canonically extends to this compactification.

(3) THE CALABI-ECKMANN MANIFOLDS

Let us take $\left(\mathbf{C}^k \times \mathbf{C}^\ell\right) \setminus \left((\mathbf{C}^k \times 0) \cup (0 \times \mathbf{C}^\ell)\right)$ and divide by the following action of $\mathbf{C}$:

$$(z_1, z_2) \longmapsto \left(A_1^\lambda z_1, A_2^\lambda z_2\right), \qquad \lambda \in \mathbf{C}.$$

A_1 and A_2 are appropriate linear operators in $\mathbf{C}^k$ and $\mathbf{C}^\ell$. For example, $A_1^\lambda = \exp \lambda$, $A_2^\lambda = \exp i\lambda$, where λ is a scalar. In the last case, the factor manifold possesses an endomorphism f which lifts to the following polynomial map

$$\mathbf{C}^k \times \mathbf{C}^\ell \to \mathbf{C}^k \times \mathbf{C}^\ell \ : \ (z_1, \ldots, z_{k+\ell}) \longmapsto (z_1^p, \ldots, z_{k+\ell}^p).$$

Recall that the Calabi-Eckmann manifolds are diffeomorphic to $\mathbf{S}^{2k-1} \times \mathbf{S}^{2\ell-1}$. The above map f has degree $d = (2(k + \ell - 1))^p$ and $h(f) = \mathrm{lov}\, f = \log d$.

(4) BLOWING UP

Let us take $W \subset V_0$ and an endomorphism $f\colon V_0 \to V_0$ such that $f^{-1}(W) = W$. The endomorphism f can be sometimes lifted to the manifold V obtained by blowing up W.

EXAMPLE. $V_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$, W is the single point $(0,0)$, and $f\colon (z_1, z_2) \mapsto (z_1^p, z_2^p)$.

(5) CONCLUDING REMARKS

A typical compact complex manifold has very few endomorphisms. For example, manifolds with nontrivial Kobayashi volume have no endomorphisms of degree ≥ 2. Do Grassmann manifolds have such endomorphisms? (No, see [3′].)

REFERENCES

[1] ARTIN, M. and B. MAZUR. On periodic points. *Ann. of Math. (2) 81* (1965), 82–99.

[1′] KALOSHIN, V. YU. and B. R. HUNT. A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms I. *Electronic Research Announcements of AMS 7* (April 18, 2001), 17–27.

[2] BOWEN, R. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc. 153* (1971), 401–414.

[3] MILNOR, J. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc. 15* (1964), 275–280.

[3′] PARANJAPE, K. H. and V. SRINIVAS. Self maps of homogeneous spaces. *Invent. Math. 98* (1989), 425–444.

[4] MISIUREWICZ, M. and F. PRZYTYCKI. Topological entropy and degree of smooth mappings. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25* (1977), 573–574.

[5] SHUB, M. Dynamical systems, filtrations and entropy. *Bull. Amer. Math. Soc. 80* (1974), 27–41.

[5′] GROMOV, M. *Partial Differential Relations.* Springer, 1986.

Mikhaïl Gromov

Institut des Hautes Études Scientifiques
35, route de Chartres
F-91440 Bures-sur-Yvette
France