5.1 About dilatation in cancellations

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$g \in f^{-1}(r)$ with $|g|_r \geq M$. We assume that g is dilated in the past after t'_0. Since the semi-flow is strongly hyperbolic, for each $n \geq 1$, in each connected component of $f^{-1}(r - nt'_0)$, there is at least one geodesic preimage $g_{-nt'_0}$ of g with $|g_{-nt'_0}|_{r-nt'_0} \geq \lambda^{nt'_0}|g|_r$. We need an estimate of the horizontal length of the other geodesic preimages of g in this stratum. Lemma 5.2 below is easily deduced from the bounded-cancellation property:

LEMMA 5.2. *With the assumptions and notation of Lemma 5.1, let $g \in f^{-1}(r)$ be some horizontal geodesic. If g^1_{-t} and g^2_{-t}, $t > 0$, are two geodesic preimages of g under σ_t which belong to a same connected component of their stratum, then $\left||g^1_{-t}|_{r-t} - |g^2_{-t}|_{r-t}\right| \leq C_{5.2}(t)$ for some constant $C_{5.2}(t)$.*

Thus, by Lemma 5.2, for any $n \geq 1$, any geodesic preimage $g_{-nt'_0}$ satisfies $|g_{-nt'_0}|_{r-nt'_0} \geq \lambda^{nt'_0}|g|_r - C_{5.2}(nt'_0)$. For $n = 1$, if $|g|_r > \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2}$, then $|g_{-t'_0}|_{r-t'_0} > 2|g|_r$. Thus, if $|g|_r > \max(M, \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2})$ then any geodesic preimage $g_{-t'_0}$ has horizontal length greater than $2|g|_r$. In particular $|g_{-t'_0}|_{r-t'_0} \geq M$ because $|g|_r > M$. By definition of a hyperbolic semi-flow, $g_{-t'_0}$ is dilated either in the future or in the past. This cannot be the case in the future since $|g_{-t'_0}|_{r-t'_0} > |g|_r$. An easy induction on n completes the proof. It suffices to set $t'_0 = (E[\max(1, \frac{\ln 2}{\ln \lambda})] + 1)t_0$ and $M' = \max(M, \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2}) + 1$. $\square$

We will assume that the constants of hyperbolicity t_0 and M are chosen to satisfy the conclusion of Lemma 5.1. Moreover the constants of hyperbolicity t_0, M, λ_+, λ_-, K are chosen large enough that computations make sense. In the sequel, we say that a path g is *C-close* to a path g' if g and g' are C-close with respect to the Hausdorff distance relative to the specified metric (the telescopic metric if none is specified). The indices of the constants refer to the lemma or proposition in which they first appear.

5.1 ABOUT DILATATION IN CANCELLATIONS

Let us recall that a *cancellation* is a horizontal geodesic whose endpoints are identified under some σ_t, $t > 0$.

LEMMA 5.3. *Let $g \in f^{-1}(r)$ be any horizontal geodesic which is dilated in the future after nt_0 for some integer $n \geq 1$. There exists a constant $C_{5.3}(n) \geq M$, which increases with n, such that if g is contained in a cancellation, then $|g|_r \leq C_{5.3}(n)$.*

Proof. Let c be the cancellation containing g. Let $c = c_1 \cup c_2$, with $[c_1]_{r+t} = [c_2]_{r+t}$ for some $t > 0$. We assume momentarily that $c_1 \cap c_2$ is an endpoint of g. The bounded-cancellation property implies that the horizontal length of a cancellation 'killed' in time t_0 (i.e. a cancellation whose pulled-tight projection after t_0 is a point) is a constant $C(t_0)$. This constant does not depend on the horizontal length of g.

Let us consider the pulled-tight image $[g]_{r+t_0}$. Let $p \subset [g]_{r+t_0}$ be the maximal subpath outside the pulled-tight image of c. This subpath p is the image of a cancellation killed at time t_0. From the observation above and the bounded-dilatation property, $|p|_{r+t_0} \leq \lambda_+^{t_0} C(t_0)$. The same arguments lead to the upper bound $(\lambda_+^{nt_0} + \lambda_+^{(n-1)t_0} + \ldots + \lambda_+^{t_0})C(t_0)$ for the horizontal length of the subpath of $[g]_{r+nt_0}$ outside $[c]_{r+nt_0}$. Since g is dilated in the future after nt_0, we have $|[g]_{r+nt_0}|_{r+nt_0} \geq \lambda^{t_0}|g|_r$. From the last two inequalities, if

$$|g|_r > \frac{(\lambda_+^{nt_0} + \lambda_+^{(n-1)t_0} + \ldots + \lambda_+^{t_0})C(t_0)}{\lambda^{t_0} - 1},$$

then the horizontal length of the subpath q of $[g]_{r+nt_0}$ in $[c]_{r+nt_0}$ is greater than $|g|_r$. If $|g|_r \geq M$, then $|q|_{r+nt_0} \geq M$ is dilated in the future after t_0 since by convention M satisfies the conclusion of Lemma 5.1. We thus obtain, for any $j \geq n$, the existence of a geodesic with horizontal length greater than $|g|_r$ in $[c]_{r+jt_0}$. This is impossible.

Let us now consider the case where $c_1 \cap c_2$ is not an endpoint of g. After some time $t > 0$, the situation will be the one described above, that is a cancellation $c' = c'_1 \cup c'_2$ with $c'_1 \cap c'_2$ an endpoint of $[g]_{r+t}$. The arguments above, together with the bounded-cancellation and bounded-dilatation properties, lead to the conclusion. $\square$

We will often encounter situations in which the pulled-tight projection of a horizontal geodesic p_1 is identified with the pulled-tight projection of another horizontal geodesic p_2 in the same stratum. In this case p_1, p_2 are not necessarily contained in cancellations. But if they lie in the same connected component of their stratum, both are contained in the union of two cancellations. Lemma 5.4 below will allow us to deal with similar situations.

Lemma 5.4. *Let p be a horizontal geodesic which admits a decomposition in r subpaths p_i such that for some constant $L \geq 0$, for any $i = 1, \ldots, r$, either $|[p_i]_{r+nt_0}|_{r+nt_0} \leq |p_i|_r$ or $L \geq |[p_i]_{r+nt_0}|_{r+nt_0} > |p_i|_r$. Then there exists a constant $C_{5.4}(n, r, L)$, which is increasing in each variable, such that if p is dilated in the future after nt_0, then $|p|_r \leq C_{5.4}(n, r, L)$.*

Proof. We set $n = 1$ in order to simplify the notation; the general case is treated in the same way. Up to permuting the indices, $|[p_i]_{r+t_0}|_{r+t_0} > |p_i|_r$ for $i = 1, \ldots, j$. Since p is dilated in the future after t_0,

$$jL + \sum_{i=j+1}^{r} |p_i|_r \geq \lambda^{t_0} \sum_{i=1}^{r} |p_i|_r .$$

Therefore $|p|_r \leq \frac{jL}{\lambda^{t_0}-1}$. □

5.2 Straight telescopic paths

Definition 5.5. A *straight* telescopic path is a telescopic path S such that if x, y are any two points in S with $x \in O^+(y) \cup O^-(y)$ then the subpath of S between x and y is equal to the orbit-segment of the semi-flow between x and y.

If S is a path containing a point x, let $S_{x,t} \subset S$ be the maximal subpath of S containing x, whose pulled-tight projection $[S_{x,t}]_{f(x)+t}$ on $f^{-1}(f(x)+t)$ is well defined. The point $\sigma_t(x)$ does not necessarily belong to $[S_{x,t}]_{f(x)+t}$. However there exists a unique point in $[S_{x,t}]_{f(x)+t}$ which minimizes the horizontal distance between $\sigma_t(x)$ and $[S_{x,t}]_{f(x)+t}$. This point is denoted by $\overline{x}_t$. Lemma 5.6 below gives an upper bound, depending on t, for the telescopic distance between x and $\overline{x}_t$.

Lemma 5.6. *Let S be any straight telescopic path. If t is any non negative real number, there exists a constant $C_{5.6}(t) \geq t$, which increases with t, such that any point $x \in S$ is at telescopic distance smaller than $C_{5.6}(t)$ from the point $\overline{x}_t$ (see above).*

Proof. If $\sigma_t(x) \in [S_{x,t}]_{f(x)+t}$, we set $C_{5.6}(t) = t$. Since S is straight, if $\sigma_t(x) \notin [S_{x,t}]_{f(x)+t}$, x belongs to a cancellation c whose endpoints lie in the past orbits of $\overline{x}_t$. The bounded-cancellation property gives an upper bound on the horizontal length of c. This leads to the conclusion. □