12. Back to mapping-telescopes

Proposition 10.1 provides a $\kappa(r,s) = Bi(A(r,s), A(r,s))$ such that this bigon is $\kappa(r,s)$-thin. Thus the given (r,s)-chain bigon is $\delta(r,s)$-thin, with $\delta(r,s) = \kappa(r,s) + 2A(r,s)$. By Lemma 11.1, the given forest-stack, which is a $(1,2)$-quasi geodesic metric space, is $2\delta(1,6)$-hyperbolic. $\square$

12. Back to mapping-telescopes

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

12.1 Statement of the theorem

An $\mathbf{R}$-*tree* (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an $\mathbf{R}$-tree is a topological tree. An $\mathbf{R}$-*forest* is a union of disjoint $\mathbf{R}$-trees.

LEMMA 12.1. *Let (Γ, d_Γ) be an $\mathbf{R}$-forest and let $\psi\colon \Gamma \to \Gamma$ be a forest-map of Γ. Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$ on K_ψ such that*

1. *The $\mathbf{R}$-forests $(f^{-1}(r), m_r)$ and $(f^{-1}(r+1), m_{r+1})$ are isometric. Each stratum $(f^{-1}(n), m_n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ).*
2. *For any real r and any horizontal geodesic $g \in f^{-1}(r)$, the map*

$$l_{r,g}\colon \begin{cases} +1 - r] \to \mathbf{R}^+ \\ \quad t \quad \mapsto |\sigma_t(g)|_{r+t} \end{cases} .$$

is monotone.

Such a horizontal metric is called a horizontal d_Γ-metric. *The telescopic metric associated to a horizontal d_Γ-metric is called a* mapping-telescope d_Γ-metric.

Proof. We make each $\Gamma \times \{n\}$, $n \in \mathbf{Z}$, an $\mathbf{R}$-forest isometric to Γ. We consider a cover of Γ by geodesics of length 1 which intersect only at their endpoints. Each $\Gamma \times \{n\}$ inherits the same cover. There is a disc $D_{e,n}$ in K_ψ for each such horizontal geodesic e in $\Gamma \times \{n\}$. This disc is bounded by e, $\psi(e)$ and the orbit-segments between the endpoints of e and those of $\psi(e)$.

We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously and monotonically, from the length of e to that of $\psi(e)$. We thus obtain a horizontal metric on the mapping-telescope. Furthermore each stratum $f^{-1}(n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ). And the maps denoted by $l_{r,g}$ in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs $D_{e,n}$ between $\Gamma \times \{n\}$ and $\Gamma \times \{n+1\}$ are copies of the discs $D_{e,n'}$ between $\Gamma \times \{n'\}$ and $\Gamma \times \{n'+1\}$, for any n, n' in $\mathbf{Z}$. This allows us to choose the horizontal metric to satisfy the further condition that $(f^{-1}(r), m_r)$ be isometric with $(f^{-1}(r+1), m_{r+1})$ for any real number r. $\square$

We now define dynamical properties for $\mathbf{R}$-forest maps.

DEFINITION 12.2. Let (Γ, d_Γ) be an $\mathbf{R}$-forest. A forest-map ψ of (Γ, d_Γ) is *weakly bi-Lipschitz* if there exist $\mu \geq 1$ and $K \geq 0$ such that $\mu d_\Gamma(x,y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x,y) - K$.

DEFINITION 12.3. Let (Γ, d_Γ) be an $\mathbf{R}$-forest. A forest-map ψ of (Γ, d_Γ) is *hyperbolic* if it is weakly bi-Lipschitz and there exist $\lambda > 1$, $N \geq 1$, $M \geq 0$ such that for any pair of points x, y in Γ with $d_\Gamma(x,y) \geq M$, either $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x,y)$ or $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x,y)$ for some x_N, y_N with $\psi^N(x_N) = x$, $\psi^N(y_N) = y$.

A hyperbolic forest-map ψ of (Γ, d_Γ) is *strongly hyperbolic* if, for any pair of points x, y with $d_\Gamma(x,y) \geq M$ and each connected component containing both a preimage of x and a preimage of y under ψ^N, there is at least one pair of such preimages x_N, y_N for which $d_\Gamma(x_N, y_N) \geq \lambda\, d_\Gamma(x,y)$.

If the forest Γ is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is

THEOREM 12.4. *Let (Γ, d_Γ) be an $\mathbf{R}$-forest. Let ψ be a strongly hyperbolic forest-map of (Γ, d_Γ) whose mapping-telescope K_ψ is connected. Then K_ψ is a Gromov-hyperbolic metric space for any mapping-telescope d_Γ-metric.*

12.2 Proof of Theorem 12.4

LEMMA 12.5. *Let* (Γ, d_Γ) *be an* $\mathbf{R}$*-forest. Let* ψ *be a weakly bi-Lipschitz forest-map of* (Γ, d_Γ). *Let* (K_ψ, f, σ_t) *be the mapping-telescope of* (ψ, Γ), *equipped with a structure of forest-stack as defined in Section 2. Then the semi-flow* $(\sigma_t)_{t\in\mathbf{R}^+}$ *is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal* d_Γ*-metric (see Lemma 12.1).*

Proof. The horizontal metric $\mathcal{H}$ agrees with the metric d_Γ on all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$ (see Lemma 12.1). Consider any horizontal geodesic g in the stratum $f^{-1}(0)$. If ψ is weakly bi-Lipschitz with constants μ_0 and K_0, then for any integer $n \geq 0$, we have $|[g]_n|_n \geq \frac{1}{\mu_0^n}|g|_0 - K_0(\frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \ldots + 1)$. Since $0 < \frac{1}{\mu_0} < 1$, the sum tends to $\frac{\mu_0}{\mu_0 - 1}$ as $n \to +\infty$. Setting $\lambda_- = \frac{1}{\mu_0}$ and $K = K_0\frac{\mu_0}{\mu_0-1}$, this proves the inequality of item (1) for horizontal geodesics in $f^{-1}(n)$, $n \in \mathbf{Z}$, and an integer time t. For the case in which t is any positive real number and $g \in f^{-1}(r)$, r any real number, just decompose $\sigma_t = \sigma_{t-E[t]} \circ \sigma_{E[t-(E[r]+1-r)]} \circ \sigma_{E[r]+1-r}$. The map σ_t is a homeomorphism from $f^{-1}(r)$ onto $f^{-1}(r+t)$ for any $t \in [0, E[r]+1-r)$. That is, for any real r, $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$ for $t \in [0, E[r]+1-r)$. The monotonicity of the maps $l_{r,g}$ (see Lemma 12.1, item (2)) implies, for any r and $t \in [0, E[r]+1-r)$, that $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0}|g|_r$. The conclusion follows. □

LEMMA 12.6. *With the assumptions and notation of Lemma 12.5, if the map* ψ *is a (strongly) hyperbolic forest-map of* (Γ, d_Γ) *then the semi-flow* $(\sigma_t)_{t\in\mathbf{R}^+}$ *is (strongly) hyperbolic with respect to any horizontal* d_Γ*-metric.*

The proof is similar to that of Lemma 12.5. □

Proof of Theorem 12.4. By Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ with horizontal metric $\mathcal{H}$ such that the semi-flow $(\sigma_t)_{t\in\mathbf{R}^+}$ is a strongly hyperbolic semi-flow with respect to $\mathcal{H}$. Hence Theorem 4.4 implies Theorem 12.4. □

13. About mapping-torus groups

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [19].