

## 2.2 Bundle gerbes

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

### Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

cohomology class in  $H^1(M, \underline{\mathrm{U}(1)}) = H^2(M, \mathbf{Z})$  defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in  $H^3(M, \mathbf{Z})$  in a similar fashion, replacing  $\mathrm{U}(1)$ -valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles  $L_{ab} \rightarrow U_a \cap U_b$  and a trivialization, i.e. unit length section,  $t_{abc}$  of the line bundle  $(\delta L)_{abc} = L_{bc}L_{ac}^{-1}L_{ab}$  over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd}t_{acd}^{-1}t_{abd}t_{abc}^{-1} = 1,$$

which makes sense since  $(\delta t)_{abcd}$  is a section of the *canonically* trivial bundle. (Each factor  $L_{ab}$  cancels with a factor  $L_{ab}^{-1}$ .) After passing to a refinement of the cover, such that all  $L_{ab}$  become trivializable, and picking trivializations,  $t_{abc}$  is simply a Čech cocycle of degree 2, hence defines a class in  $H^2(M, \underline{\mathrm{U}(1)}) = H^3(M, \mathbf{Z})$ . The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if  $M$  is a connected, oriented 3-manifold, the generator of  $H^3(M, \mathbf{Z}) = \mathbf{Z}$  can be described in terms of the cover  $U_1, U_2$ , where  $U_1$  is an open ball around a given point  $p \in M$ , and  $U_2 = M \setminus \{p\}$ , using the degree one line bundle over  $U_1 \cap U_2 \cong S^2 \times (0, 1)$ .

## 2.2 BUNDLE GERBES

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let  $\pi: X \rightarrow M$  be a fiber bundle, or more generally a surjective submersion. (Different components of  $X$  may have different dimensions.) For each  $k \geq 0$  let  $X^{[k]}$  denote the  $k$ -fold fiber product of  $X$  with itself. There are  $k+1$  projections  $\partial^i: X^{[k+1]} \rightarrow X^{[k]}$ , omitting the  $i$ th factor in the fiber product. Suppose we are given a smooth function  $\chi: X^{[2]} \rightarrow \mathrm{U}(1)$ , satisfying a cocycle condition  $\delta\chi = 1$  where

$$\delta\chi := \partial_0^* \chi \partial_1^* \chi^{-1} \partial_2^* \chi: X^{[3]} \rightarrow \mathrm{U}(1).$$

Then  $\chi$  determines a Hermitian line bundle  $L \rightarrow M$ , with fibers at  $m \in M$  the space of all linear maps  $\phi: X_m = \pi^{-1}(m) \rightarrow \mathbf{C}$  such that  $\phi(x) = \chi(x, x')\phi(x')$ . Given local sections  $\sigma_a: U_a \rightarrow X$  of  $X$ , the pull-backs of  $\chi$  under the maps  $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$  give transition functions  $\chi_{ab}$  for the line bundle.

Again, replacing  $\mathrm{U}(1)$ -valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle  $L \rightarrow X^{[2]}$  and a trivializing section  $t$  of the line bundle  $\delta L = \partial_0^* L \otimes \partial_1^* L^{-1} \otimes \partial_2^* L$

over  $X^{[3]}$ , satisfying a compatibility condition  $\delta t = 1$  over  $X^{[4]}$  (which makes sense since  $\delta t$  is a section of the canonically trivial bundle  $\delta\delta L$ ). Given local sections  $\sigma_a: U_a \rightarrow X$ , one can pull these data back under the maps  $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$  and  $(\sigma_a, \sigma_b, \sigma_c): U_a \cap U_b \cap U_c \rightarrow X^{[3]}$  to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of  $(X, L, t)$  is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with  $X$  the disjoint union of the sets  $U_a$  in the given cover.

**REMARK 2.1.** In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called ‘locally split’) is used that every point  $x \in M$  admits an open neighborhood  $U$  and a map  $\sigma: U \rightarrow X$  such that  $\pi \circ \sigma = \text{id}$ . However, this condition seems insufficient in the smooth category, as the fiber product  $X \times_M X$  need not be a manifold unless  $\pi$  is a submersion.

### 2.3 SIMPLICIAL GERBES

Murray’s construction fits naturally into a wider context of *simplicial gerbes*. We refer to Mostow-Perchik’s notes of lectures by R. Bott [23] and to Dupont’s paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold*  $M_\bullet$  is a sequence of manifolds  $(M_n)_{n=0}^\infty$ , together with *face maps*  $\partial_i: M_n \rightarrow M_{n-1}$  for  $i = 0, \dots, n$  satisfying relations  $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$  for  $i < j$ . (The standard definition also involves *degeneracy maps* but these need not concern us here.) The (*fat*) *geometric realization* of  $M_\bullet$  is the topological space  $\|M\| = \coprod_{n=1}^\infty \Delta^n \times M_n / \sim$ , where  $\Delta^n$  is the  $n$ -simplex and the relation is  $(t, \partial_i(x)) \sim (\partial^i(t), x)$ , for  $\partial^i: \Delta^{n-1} \rightarrow \Delta^n$  the inclusion as the  $i$ th face. A (smooth) simplicial map between simplicial manifolds  $M_\bullet, M'_\bullet$  is a collection of smooth maps  $f_n: M_n \rightarrow M'_n$  intertwining the face maps; such a map induces a map between the geometric realizations.

### EXAMPLES 2.2.

(a) If  $S$  is any manifold, one can define a simplicial manifold  $E_\bullet S$  where  $E_n S$  is the  $n+1$ -fold cartesian product of  $S$ , and  $\partial_j$  omits the  $j$ th factor. It is known [23] that the geometric realization  $\|ES\|$  of this simplicial manifold is contractible. More generally, if  $X \rightarrow M$  is a fiber bundle with fiber  $S$ ,