3. The classification of unoriented rational knots

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

3. THE CLASSIFICATION OF UNORIENTED RATIONAL KNOTS

In this section we shall prove Schubert's theorem for unoriented rational knots. It is convenient to say that reduced fractions p/q and p'/q' are arithmetically equivalent, written $p/q \sim p'/q'$, if $p = p'$ and either $qq' \equiv 1 \mod p$ or $q \equiv q' \mod p$. We shall call two rational tangles *arithmetically equivalent* if their fractions are arithmetically equivalent. In this language, Schubert's theorem states that two unoriented rational tangles close to form isotopic knots if and only if they are arithmetically equivalent.

We only need to consider numerator closures of rational tangles, since the denominator closure of a tangle T is simply the numerator closure of its rotate $-\frac{1}{T}$. From the discussions in Section 2 a rational tangle may be assumed to be in continued fraction form and by Remark 1, *the length* of a rational tangle may be assumed to be *odd.* A rational knot is said to be *in standard form, in continued fraction form, alternating or in canonical form* if it is the numerator closure of a rational tangle that is in standard form, in continued fraction form, alternating or in canonical form respectively. By the alternating property of rational knots we may assume all rational knot diagrams to be *alternating*. The diagrams and the isotopies of the rational knots are meant to take place in the 2-sphere and not in the plane.

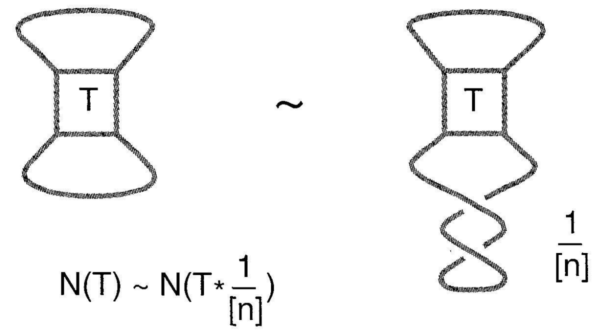


FIGURE 12
Twisting the bottom of a tangle

BOTTOM TWISTS. The simplest instance of two rational tangles being non-isotopic but having isotopic numerators is adding a number of twists at the bottom of a tangle, see Figure 12. Indeed, let T be a rational tangle and let $T * 1/[n]$ be the tangle obtained from T by adding n bottom twists, for any $n \in \mathbf{Z}$. We have $N(T * 1/[n]) \sim N(T)$, but $F(T * 1/[n]) = F(1/([n] + 1/T)) = 1/(n + 1/F(T))$; so, if

$$F(T) = p/q,$$

then

$$F(T * 1/[n]) = p/(np+q),$$

thus the two tangles are not isotopic. If we set $np+q=q'$ we have $q \equiv q'$ mod p, just as Theorem 2 predicts.

Reducing all possible bottom twists of a rational tangle yields a rational tangle with fraction $\frac{P}{Q}$ such that

$$|P| > |Q|.$$

To see this, suppose that we are dealing with $\frac{P}{Q'}$ with $P < Q'$ and both P and Q' positive (we leave it to the reader to fill in the details for Q' negative). Then

$$\frac{P}{Q'} = \frac{1}{\frac{Q'}{P}} = \frac{1}{n+\frac{Q}{P}} = \frac{1}{n+\frac{1}{P/Q}},$$

where

$$Q' = nP + Q \equiv Q \mod P,$$

for n and Q positive and $Q < P$. So, by the Conway Theorem, the rational tangle $[\frac{P}{Q'}]$ differs from the tangle $[\frac{P}{Q}]$ by n bottom twists, and so $N([\frac{P}{Q'}]) \sim N([\frac{P}{Q}])$. Figure 13 illustrates an example of this arithmetic. Note that a tangle with fraction $\frac{P}{Q}$ such that $|P| > |Q|$ always ends with a number of horizontal twists. So, if $T = [[a_1], [a_2], \dots, [a_n]]$ then $a_1 \neq 0$. If T is in twist form then T will not have any top or bottom twists. We shall say that a rational tangle whose fraction satisfies the above inequality is in *reduced form.*

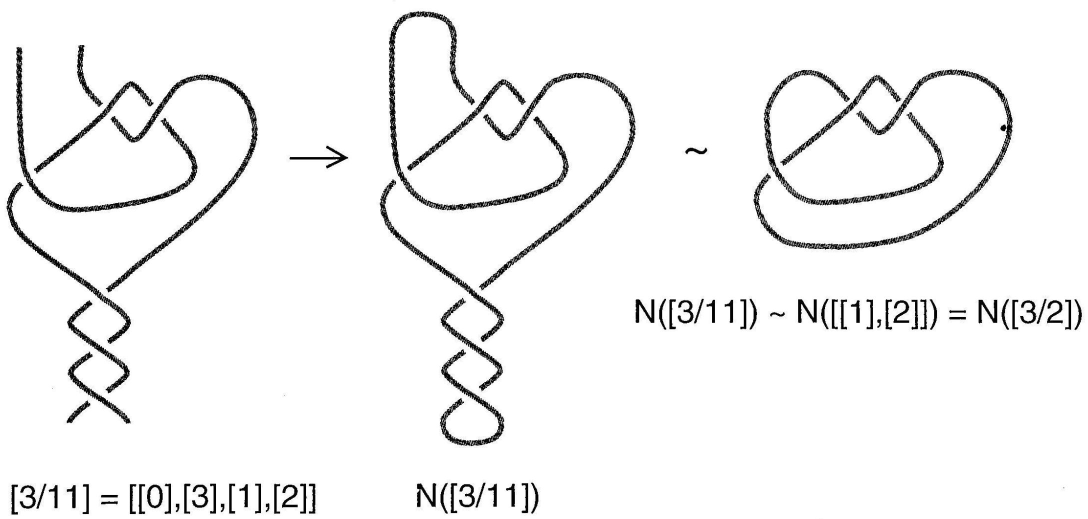


FIGURE 13
Reducing the bottom twists

The proof of Theorem 2 now proceeds in two stages. First, (in 3.1) we look for all possible places where we could cut a rational knot K open to a rational tangle, and we show that all cuts that open K to other rational tangles give tangles arithmetically equivalent to the tangle T. Second, (in 3.2) given two isotopic reduced alternating rational knot diagrams, we have to check that the rational tangles that they open to are arithmetically equivalent. By the solution to the Tait Conjecture these isotopic knot diagrams will differ by a sequence of flypes. So we analyze what happens when a flype is performed on K.

3.1 THE CUTS

Let K be a rational knot that is the numerator closure of a rational tangle T. We will look for all 'rational' cuts on K. In our study of cuts we shall assume that T is in reduced canonical form. The more general case where T is in reduced alternating twist form is completely analogous and we make a remark at the end of the subsection. Moreover, the cut analysis in the case where $a_1 = 0$ is also completely analogous for all cuts with appropriate adjustments. There are three types of rational cuts.

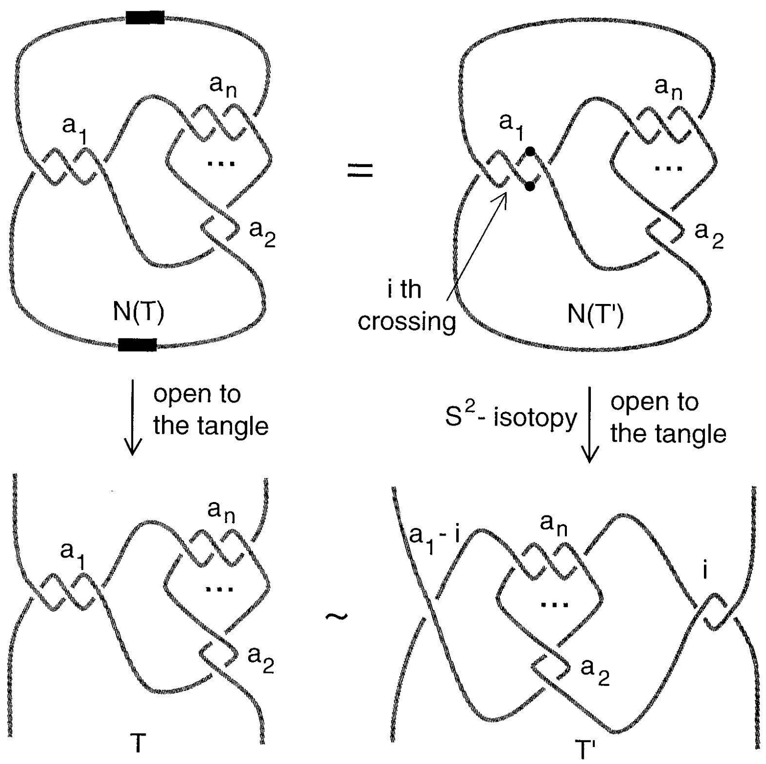
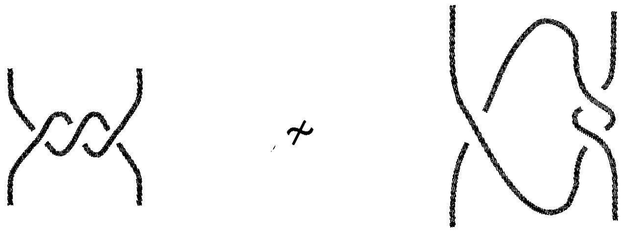


FIGURE 14

Standard cuts

THE STANDARD CUTS. The tangle $T = [[a_1], [a_2], \ldots, [a_n]]$ is said to arise as *the standard cut* on $K = N(T)$. If we cut K at another pair of 'vertical' points that are adjacent to the ith crossing of the elementary tangle $[a_1]$ (counting from the outside towards the inside of T) we obtain the alternating rational tangle in twist form $T' = [[a_1 - i], [a_2], \ldots, [a_n]] + [i]$. Clearly, this tangle is isotopic to T by a sequence of flypes that send all the horizontal twists to the right of the tangle. See the right hand illustration of Figure 14 for $i = 2$. Thus, by the Conway Theorem, T' will have the same fraction as T. Any such cut on K shall be called a *standard cut* on K.

THE SPECIAL CUTS. A key example of the arithmetic relationship of the classification of rational knots is illustrated in Figure 15. The two tangles $T = [-3]$ and $S = [1] + \frac{1}{[2]}$ are non-isotopic by the Conway Theorem, since $F(T) = -3 = 3/-1$, while $F(S) = 1 + 1/2 = 3/2$. But they have isotopic numerators: $N(T) \sim N(S)$, the left-handed trefoil. Now $-1 \equiv 2 \mod 3$, confirming Theorem 2.



$T = [-3]$ $\qquad$ $S = [1] + \frac{1}{[2]}$

FIGURE 15
An example of the special cut

We now analyze the above example in general. Let $K = N(T)$, where $T = [[a_1], [a_2], \ldots, [a_n]]$. Since T is assumed to be in reduced form, it follows that $a_1 \neq 0$, so T can be written in the form $T = [+1] + R$ or $T = [-1] + R$, and the tangle R is also rational.

The indicated horizontal crossing $[+1]$ of the tangle $T = [+1] + R$, which is the first crossing of $[a_1]$ and the last created crossing of T, may also be seen as a vertical one. So, instead of cutting the diagram K open at the two standard cutpoints to obtain the tangle T, we cut at the two other marked 'horizontal' points on the first crossing of the subtangle $[a_1]$ to obtain a new 2-tangle T' (see Figure 16). T' is clearly rational, since R is rational. The tangle T' is said to arise as *the special cut* on K.

We would like to identify this rational tangle T'. For this reason we first swing the upper arc of K down to the bottom of the diagram in order to free the region of the cutpoints. By our convention for the signs of crossings in

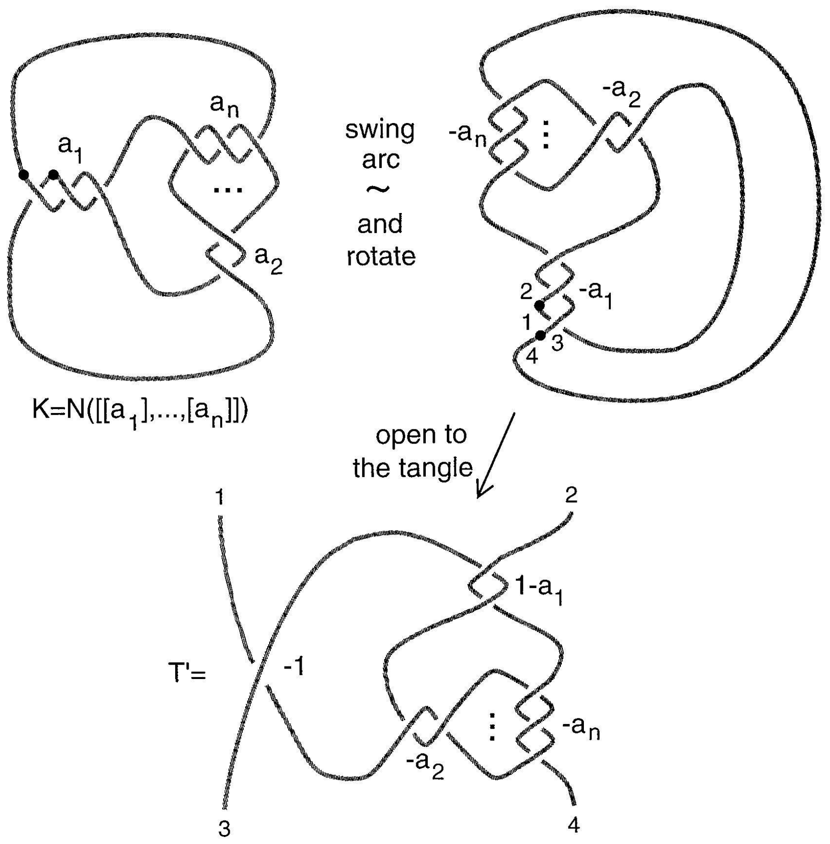


FIGURE 16
Preparing for the special cut

terms of the checkerboard shading, this forces all crossings of T to change sign from positive to negative and vice versa. We then rotate K by $90°$ on its plane (see right-hand illustration of Figure 16). This forces all crossings of T to change from horizontal to vertical and vice versa. In particular, the marked crossing $[+1]$, that was seen as a vertical one in T, will now look as a horizontal $[-1]$ in T'. In fact, this will be the only last horizontal crossing of T', since all other crossings of $[a_1]$ are now vertical. So, if $T = [[a_1], [a_2], \ldots, [a_n]]$ then $R = [[a_1 - 1], [a_2], \ldots, [a_n]]$ and

$$T' = [[-1], [1 - a_1], [-a_2], \ldots, [-a_n]] .$$

Note that if the crossings of K were all of negative type, thus all the a_i's would be negative, the tangle T' would be $T' = [[+1], [-1 - a_1], [-a_2], \ldots, [-a_n]]$. In the example of Figure 15 if we took $R = [-2]$, then $T = [-1] + R$ and $T' = S = [[+1], [+2]]$.

The special cut is best illustrated in Figure 17. We consider the rational knot diagram $K = N([+1] + R)$. (We analyze $N([-1] + R)$ in the same way.) As we see here, a sequence of isotopies and cutting K open allow us to read the new tangle:

$$T' = [-1] - \frac{1}{R} .$$

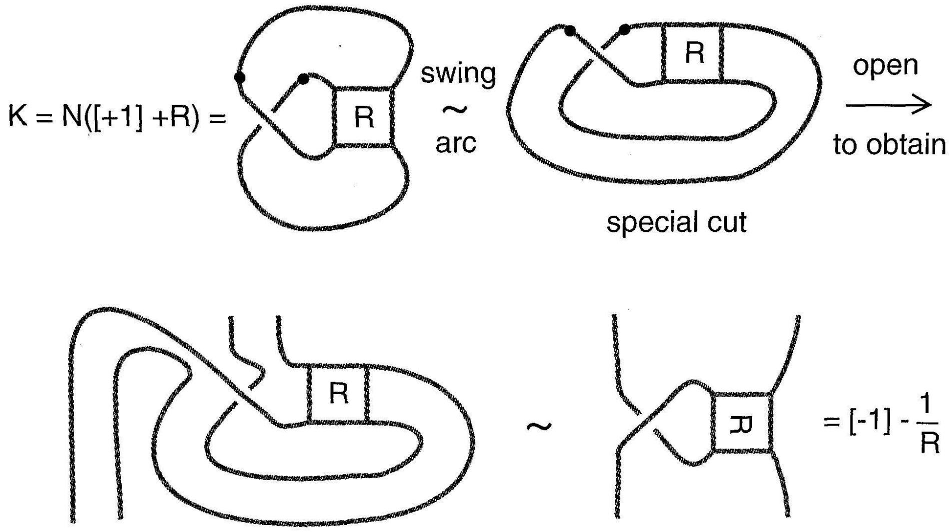


FIGURE 17
The tangle of the special cut

From the above we have $N([+1]+R) \sim N([-1]-\frac{1}{R})$. Let now the fractions of T,R and T' be $F(T) = P/Q$, $F(R) = p/q$ and $F(T') = P'/Q'$ respectively. Then

$$F(T) = F([+1]+R) = 1 + p/q = (p+q)/q = P/Q,$$

while

$$F(T') = F([-1]-1/R) = -1 - q/p = (p+q)/(-p) = P'/Q'.$$

The two fractions are different, thus the two rational tangles that give rise to the same rational knot are not isotopic. We observe that $P = P'$ and

$$q \equiv -p \mod (p+q) \iff Q \equiv Q' \mod P.$$

This arithmetic equivalence demonstrates another case for Theorem 2. Notice that, although both the bottom twist and the special cut fall into the same arithmetic equivalence, the arithmetic of the special cut is more subtle than the arithmetic of the bottom twist.

If we cut K at the two lower horizontal points of the first crossing of $[a_1]$ we obtain the same rational tangle T'. Also, if we cut at any other pair of upper or lower horizontal adjacent points of the subtangle $[a_1]$ we obtain a rational tangle in twist form isotopic to T'. Such a cut shall be called a *special cut*. See Figure 18 for an example. Finally, we may cut K at any pair of upper or lower horizontal adjacent points of the subtangle $[a_n]$. We shall call this a *special palindrome cut.* We will discuss this case after having analyzed the last type of a cut, the palindrome cut.

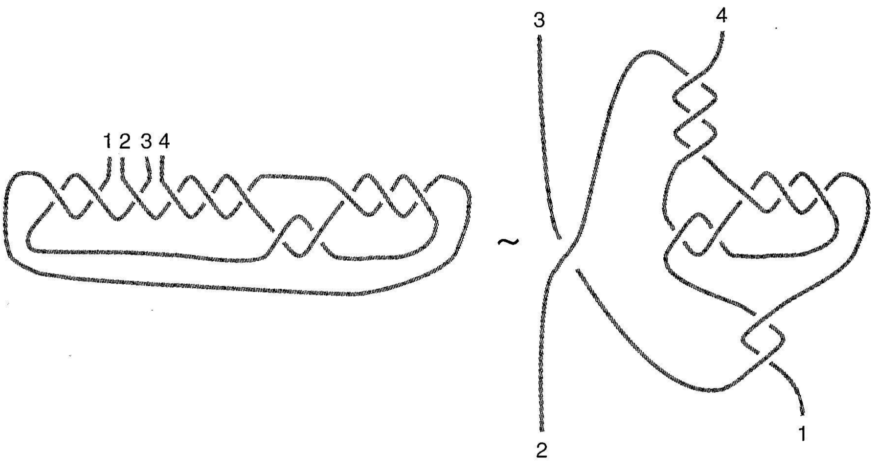


FIGURE 18
A special cut

NOTE. We would like to point out that the horizontal-vertical ambiguity of the last crossing of a rational tangle $T = [[a_1], \ldots, [a_{n-1}], [a_n]]$, which with the special cut on $K = N(T)$ gives rise to the tangle $[[\mp 1], [\pm 1 - a_1], [-a_2], \ldots, [-a_n]]$, is very similar to the horizontal-vertical ambiguity of the first crossing that does not change the tangle and it gives rise to the tangle continued fraction $[[a_1], \ldots, [a_{n-1}], [a_n \mp 1], [\pm 1]]$.

REMARK 2. A special cut is nothing more than the addition of a bottom twist. Indeed, as Figure 19 illustrates, applying a positive bottom twist to the tangle T' of the special cut yields the tangle $S = ([-1] - 1/R) * [+1]$, and we find that if $F(R) = p/q$ then $F([+1] + R) = (p + q)/q$ while $F(([-1] - 1/R) * [+1]) = 1/(1 + 1/(-1 - q/p)) = (p + q)/q$. Thus we see that the fractions of $T = [+1] + R$ and $S = ([-1] - 1/R) * [+1]$ are equal and by the Conway Theorem the tangle S is isotopic to the original tangle T of the standard cut. The isotopy move is nothing but the transfer move of Figure 11. The isotopy is illustrated in Figure 19. Here we used the Flipping Lemma.

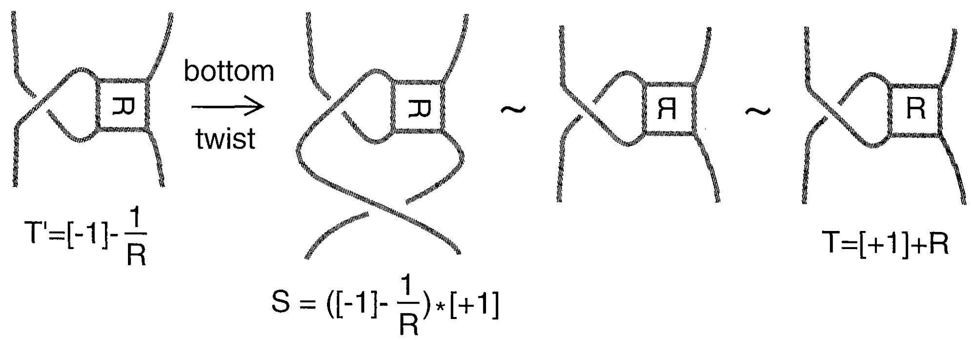


FIGURE 19
Special cuts and bottom twists

THE PALINDROME CUTS. In Figure 20 we see that the tangles

$$T = [[2],[3],[4]] = [2] + \frac{1}{[3] + \frac{1}{[4]}}$$

and

$$S = [[4],[3],[2]] = [4] + \frac{1}{[3] + \frac{1}{[2]}}$$

both have the same numerator closure. This is another key example of the basic relationship given in the classification of rational knots.

In the general case if $T = [[a_1],[a_2],\ldots,[a_n]]$, we shall call the tangle $S = [[a_n],[a_{n-1}],\ldots,[a_1]]$ *the palindrome of* T. Clearly these tangles have the same numerator: $K = N(T) = N(S)$. Cutting open K to yield T is the standard cut, while cutting to yield S shall be called *the palindrome cut* on K.

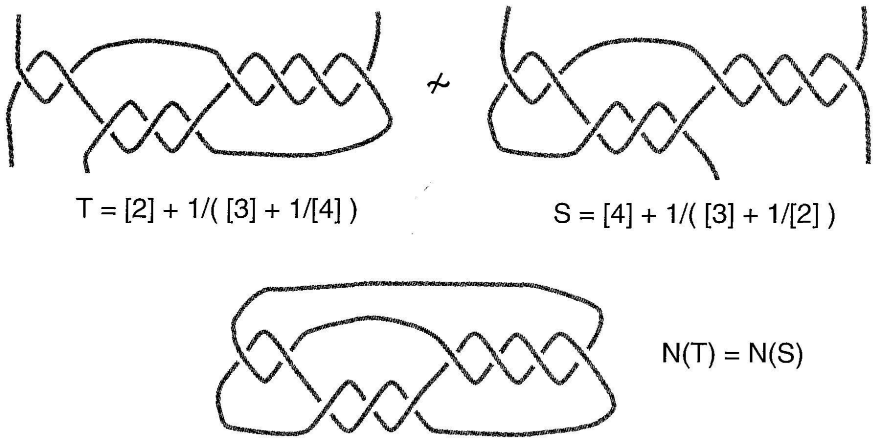


FIGURE 20
An instance of the palindrome equivalence

The tangles in Figure 20 have corresponding fractions

$$F(T) = 2 + \frac{1}{3 + \frac{1}{4}} = \frac{30}{13} \quad \text{and} \quad F(S) = 4 + \frac{1}{3 + \frac{1}{2}} = \frac{30}{7}.$$

Note that $7 \cdot 13 \equiv 1 \mod 30$. This is the other instance of the arithmetic behind the classification of rational knots in Theorem 2. In order to check the arithmetic in the general case of the palindrome cut we need to generalize this pattern to arbitrary continued fractions and their palindromes (obtained by reversing the order of the terms). Then we have the following

THEOREM 4 (Palindrome Theorem). *Let* $\{a_1, a_2, \ldots, a_n\}$ *be a collection of* n *non-zero integers, and let* $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ *and* $\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$. *Then* $P = P'$ *and* $QQ' \equiv (-1)^{n+1} \mod P$.

The Palindrome Theorem is a known result about continued fractions. For example see [35] or [16], p. 25, Exercise 2.1.9. We shall give here our proof of this statement. For this we will first present a way of evaluating continued fractions via 2×2 matrices (compare with [11], [18]). This method of evaluation is crucially important in our work in the rest of the paper. Let $\frac{p}{q} = [a_2, a_3, \ldots, a_n]$. Then we have:

$$[a_1, a_2, \ldots, a_n] = a_1 + \frac{1}{\frac{p}{q}} = a_1 + \frac{q}{p} = \frac{a_1 p + q}{p} = \frac{p'}{q'}.$$

Taking the convention that $\left[\binom{p}{q}\right] := \frac{p}{q}$, with our usual conventions for formal fractions such as $\frac{1}{0}$, we can thus write a corresponding matrix equation in the form

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a_1 p + q \\ p \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

We let

$$M(a_i) = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrices $M(a_i)$ are said to be the *generating matrices* for continued fractions, as we have:

LEMMA 1 (Matrix interpretation for continued fractions). *For any sequence of non-zero integers $\{a_1, a_2, \ldots, a_n\}$ the value of the corresponding continued fraction is given through the following matrix product*

$$[a_1, a_2, \ldots, a_n] = [M(a_1)M(a_2) \cdots M(a_n) \cdot v]$$

where

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proof. We observe that

$$\left[M(a_n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} a_n \\ 1 \end{pmatrix}\right] = a_n = [a_n]$$

and

$$\left[M(a_{n-1}) \begin{pmatrix} a_n \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} a_{n-1}a_n + 1 \\ a_n \end{pmatrix}\right] = [a_{n-1}, a_n].$$

Now the lemma follows by induction. □

Proof of the palindrome theorem. We wish to compare $\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$ and $\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1]$. By Lemma 1 we can write

$$\frac{P}{Q} = [M(a_1)M(a_2)\cdots M(a_n) \cdot v] \quad \text{and} \quad \frac{P'}{Q'} = [M(a_n)M(a_{n-1})\cdots M(a_1) \cdot v] .$$

Let

$$M = M(a_1)M(a_2)\cdots M(a_n)$$

and

$$M' = M(a_n)M(a_{n-1})\cdots M(a_1) .$$

Then $\frac{P}{Q} = [M \cdot v]$ and $\frac{P'}{Q'} = [M' \cdot v]$. We observe that

$$\begin{aligned} M^T &= (M(a_1)M(a_2)\cdots M(a_n))^T = (M(a_n))^T(M(a_{n-1}))^T \cdots (M(a_1))^T \\ &= M(a_n)M(a_{n-1})\cdots M(a_1) = M' , \end{aligned}$$

since $M(a_i)$ is symmetric, where M^T is the transpose of T. Thus

$$M' = M^T .$$

Let

$$M = \begin{pmatrix} X & Y \\ Z & U \end{pmatrix} .$$

In order that the equations $[M \cdot v] = \frac{P}{Q}$ and $[M^T \cdot v] = \frac{P'}{Q'}$ are satisfied it is necessary that $X = P$, $X = P'$, $Z = Q$ and $Y = Q'$. That is, we should have:

$$M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} P & Q \\ Q' & U \end{pmatrix} .$$

Furthermore, since the determinant of $M(a_i)$ is equal to -1, we have that

$$\det(M) = (-1)^n .$$

Thus

$$PU - QQ' = (-1)^n ,$$

so that

$$QQ' \equiv (-1)^{n+1} \mod P ,$$

and the proof of the Theorem is complete. □

REMARK 3. Note in the argument above that the entries of the matrix $M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$ of a given continued fraction $[a_1, a_2, \ldots, a_n] = \frac{P}{Q}$ involve also the evaluation of its palindrome $[a_n, a_{n-1}, \ldots, a_1] = \frac{P}{Q'}$.

Returning now to the analysis of the palindrome cut, we apply Theorem 4 in order to evaluate the fraction of palindrome rational tangles $T = [\frac{P}{Q}]$ and $S = [\frac{P'}{Q'}]$. From the above analysis we have $P = P'$. Also, by our assumption these tangles have continued fraction forms with odd length n, so we have the relation

$$QQ' \equiv 1 \mod P$$

and this is the second of the arithmetic relations of Theorem 2.

If we cut $K = N(T)$ at any other pair of 'vertical' points of the subtangle $[a_n]$ we obtain a rational tangle in twist form isotopic to the palindrome tangle S. Any such cut shall be called a *palindrome cut.*

Having analyzed the arithmetic of the palindrome cuts we can now return to the special palindrome cuts on the subtangle $[a_n]$. These may be considered as special cuts on the palindrome tangle S. So, the fraction of the tangle of such a cut will satisfy the first type of arithmetic relation of Theorem 2 with the fraction of S, namely a relation of the type $q \equiv q' \mod p$, which, consequently, satisfies the second type of arithmetic relation with the fraction of T, namely a relation of the type $qq' \equiv 1 \mod p$. In the end a special palindrome cut will satisfy an arithmetic relation of the second type. This concludes the arithmetic study of the rational cuts.

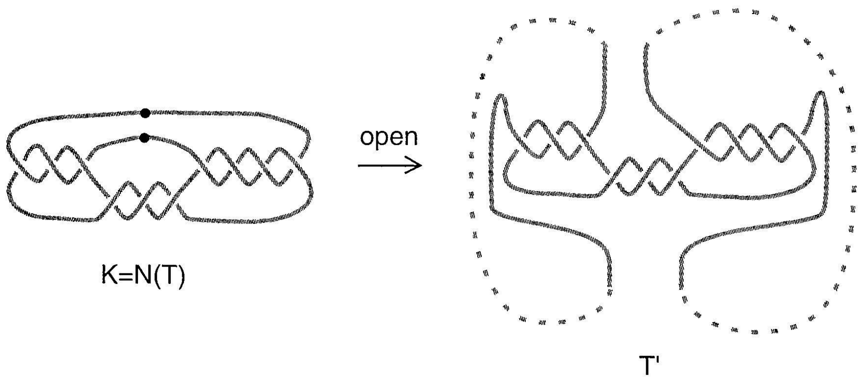


FIGURE 21
A non-rational cut

We now claim that the above listing of the three types of rational cuts is a complete catalog of cuts that can open the link K to a rational tangle: the standard cuts, the special cuts and the palindrome cuts. This is the crux of our proof.

In Figure 21 we illustrate one example of a cut that is not rational. This is a possible cut made in the middle of the representative diagram $N(T)$. Here we see that if T' is the tangle obtained from this cut, so that $N(T') = K$, then $D(T')$ is a connected sum of two non-trivial knots. Hence the denominator $K' = D(T')$ is not prime. Since we know that both the numerator and the denominator of a rational tangle are prime (see [5], p. 91 or [19], Chapter 4, pp. 32–40), it follows that T' is not a rational tangle. Clearly the above argument is generic. It is not hard to see by enumeration that all possible cuts with the exception of the ones we have described will not give rise to rational tangles. We omit the enumeration of these cases.

This completes the proof that all of the rational tangles that close to a given standard rational knot diagram are arithmetically equivalent.

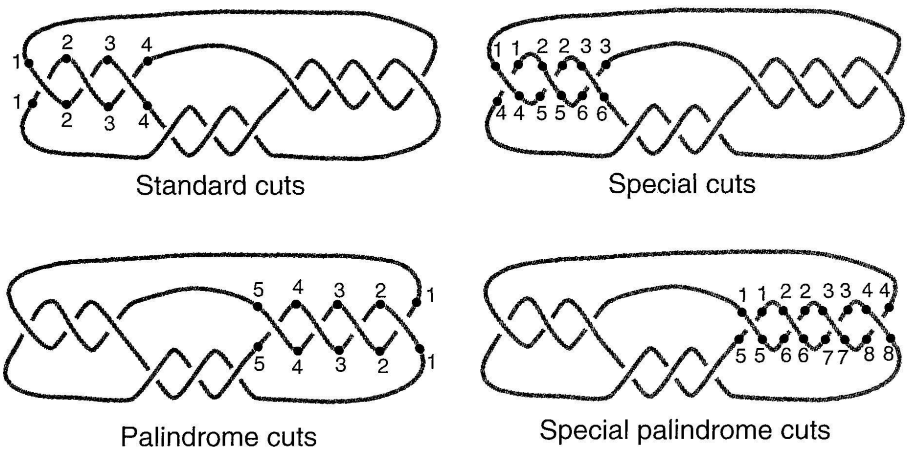


FIGURE 22
Standard, special, palindrome and special palindrome cuts

In Figure 22 we illustrate on a representative rational knot in 3-strand-braid form all the cuts that exhibit that knot as a closure of a rational tangle. Each pair of points is marked with the same number. •

REMARK 4. It follows from the above analysis that if T is a rational tangle in twist form, which is isotopic to the standard form $[[a_1], [a_2], \ldots, [a_n]]$, then all arithmetically equivalent rational tangles can arise by any cut of the above types either on the crossings that add up to the subtangle $[a_1]$ or on the crossings of the subtangle $[a_n]$.

3.2 THE FLYPES

Diagrams for knots and links are represented on the surface of the two-sphere, S^2, and then notationally on a plane for purposes of illustration.

Let $K = N(T)$ be a rational link diagram with T a rational tangle in twist form. By an appropriate sequence of flypes (recall Definition 1) we may assume, without loss of generality, that T is alternating and in continued fraction form, i.e. T is of the form $T = [[a_1], [a_2], \ldots, [a_n]]$ with the a_i's all positive or all negative. From the ambiguity of the first crossing of a rational tangle we may assume that n is odd. Moreover, from the analysis of the bottom twists in the previous subsection we may assume that T is in reduced form. Then the numerator $K = N(T)$ will be a reduced alternating knot diagram. This follows from the primality of K.

Let K and K' be two isotopic, reduced, alternating rational knot diagrams. By the Tait Conjecture they will differ by a finite sequence of flypes. In considering how rational knots can be cut open to produce rational tangles, we will examine how the cuts are affected by flyping. We analyze all possible flypes to prove that it is sufficient to consider the cuts on a single alternating reduced diagram for a given rational knot K. Hence the proof will be complete at that point. We need first two definitions and an observation about flypes.

DEFINITION 3. We shall call *region of a flype* the part of the knot diagram that contains precisely the subtangle and the crossing that participate in the flype. The region of a flype can be enclosed by a simple closed curve on the plane that intersects the tangle in four points.

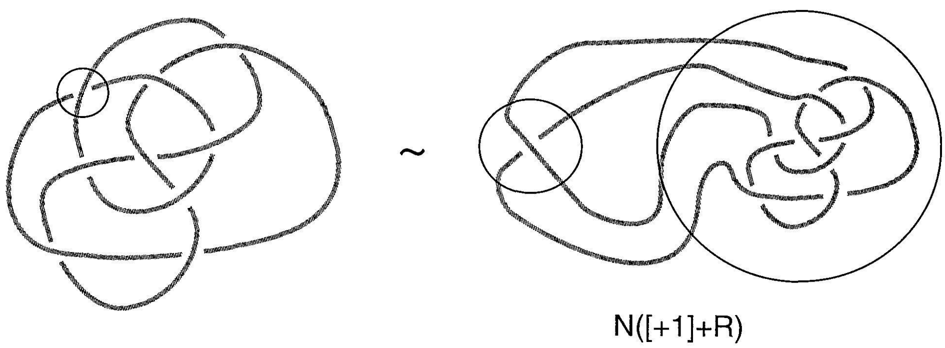


FIGURE 23
Decomposing into $N([\pm 1] + R)$

DEFINITION 4. A *pancake flip* of a knot diagram in the plane is an isotopy move that rotates the diagram by $180°$ in space around a horizontal or vertical axis on its plane and then it replaces it on the plane. Note that any knot diagram in S^2 can be regarded as a knot diagram in a plane.

In fact, the pancake flip is actually obtained by flypes so long as we allow as background moves isotopies of the diagram in S^2. To see this, note as in Figure 23 that we can assume without loss of generality that the diagram in question is of the form $N([\pm 1] + R)$ for some tangle R not necessarily rational. (Isolate one crossing at the 'outer edge' of the diagram in the plane and decompose the diagram into this crossing and a complementary tangle, as shown in Figure 23.) In order to place the diagram in this form we only need to use isotopies of the diagram in the plane.

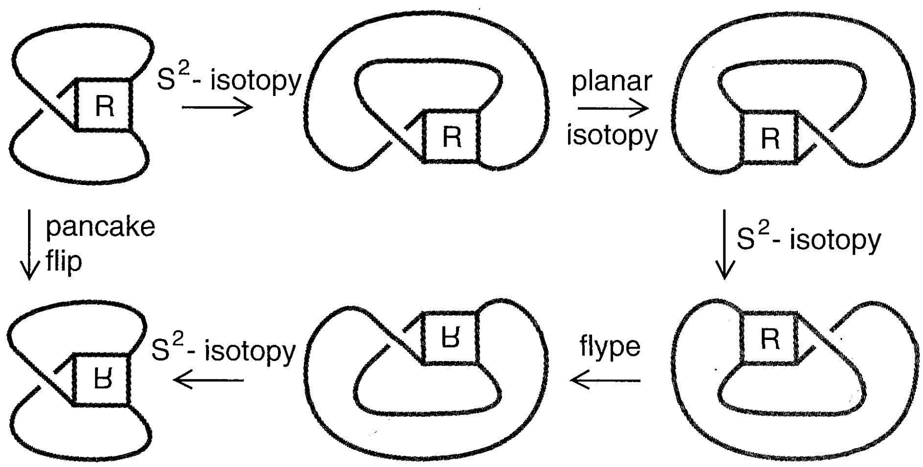


FIGURE 24
Pancake flip

Note now, as in Figure 24, that the pancake flip applied to $N([\pm 1] + R)$ yields a diagram that can be obtained by a combination of a planar isotopy, S^2-isotopies and a flype. (By an S^2*-isotopy* we mean the sliding of an arc around the back of the sphere.) This is valid for R any 2-tangle. We will use this remark in our study of rational knots and links.

We continue with a general remark about the form of a flype in any knot or link in S^2. View Figure 25. First look at parts A and B of this figure. Diagram A is shown as a composition of a crossing and two tangles P and Q. Part B is obtained from a flype of part A, where the flype is applied to the crossing in conjunction with the tangle P. This is the general pattern of the application of a flype. The flype is applied to a composition of a crossing with a tangle, while the rest of the diagram can be regarded as contained within a second tangle that is left fixed under the flyping.

Now look at diagrams C and D. Diagram D is obtained by a flype using Q and a crossing on diagram C. But diagram C is isotopic by a planar isotopy

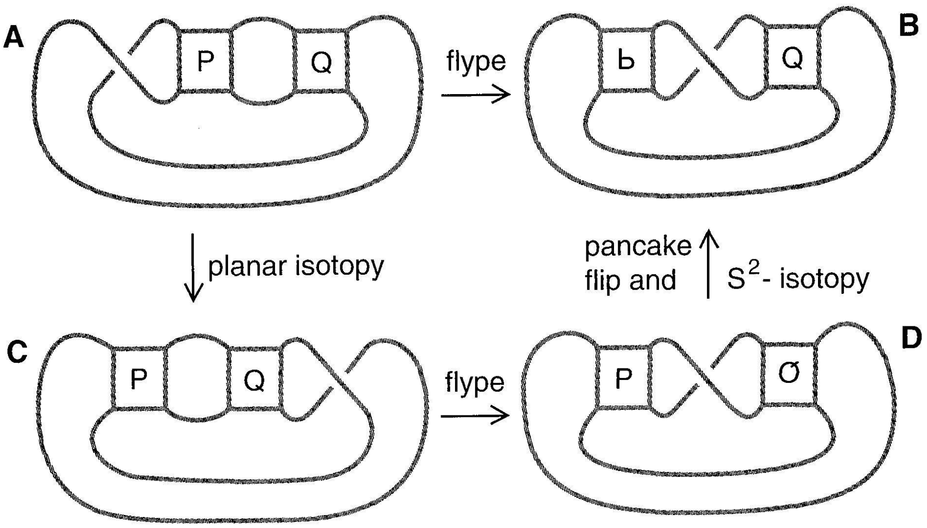


FIGURE 25
The complementary flype

to diagram A, and diagrams B and D are related by a pancake flip (combined with an isotopy that swings two arcs around S^2). Thus we see that:

Up to a pancake flip one can choose to keep either of the tangles P or Q fixed in performing a flype.

Let now $K = N(T)$ and $K' = N(T')$ be two reduced alternating rational knot diagrams that differ by a flype. The rational tangles T and T' are in reduced alternating twist form and without loss of generality T may be assumed to be in continued fraction form. Then, recall from Section 2 that the region of the flype on K can either include a rational truncation of T or some crossings of a subtangle $[a_i]$, see Figure 26. In the first case the two subtangles into which K decomposes are both rational and each will be called the *complementary tangle* of the other. In the second case the flype has really trivial effect and the complementary tangle is not rational, unless $i = 1$ or n.

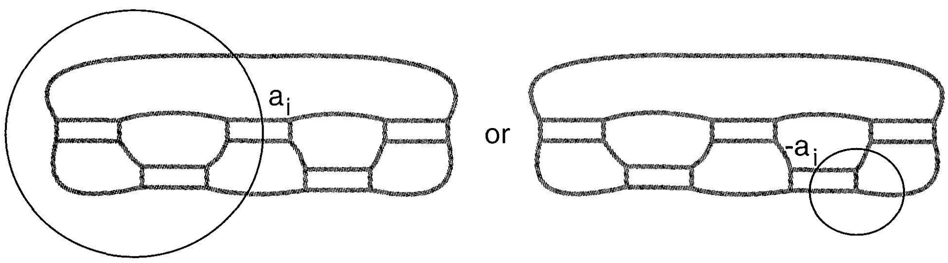


FIGURE 26
Flypes of rational knots

For the cutpoints of T on $K = N(T)$ there are three possibilities:

1. they are outside the region of the flype,
2. they are inside the flyped subtangle,
3. they are inside the region of the flype and outside the flyped subtangle.

If the cutpoints are outside the region of the flype, then the flype is taking place inside the tangle T and so there is nothing to check, since the new tangle is isotopic and thus arithmetically equivalent to T.

We concentrate now on the first case of the region of a flype. If the cutpoints are inside the flyped subtangle then, by Figure 25, this flype can be seen as a flype of the complementary tangle followed by a pancake flip. The region of the flype of the complementary tangle does not contain the cut points, so it is a rational flype that isotopes the tangle to itself. The pancake flip also does not affect the arithmetic, because its effect on the level of the tangle T is simply a horizontal or a vertical flip, and we know that a flipped rational tangle is isotopic to itself.

If the region of the flype encircles a number of crossings of some $[a_i]$ then the cutpoints cannot lie in the region, unless $i = 1$ or n. If the cutpoints do not lie in the region of the flype, there is nothing to check. If they do, then the complementary tangle is isotopic to T, and the pancake flip produces an isotopic tangle.

Finally, if the cutpoints are inside the region of the flype and outside the flyped subtangle, i.e. they are near the crossing of the flype, then there are three cases to check. These are illustrated in Figure 27.

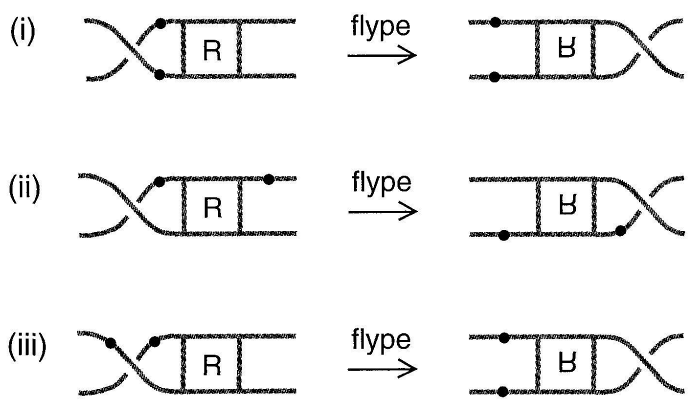


FIGURE 27
Flype and cut interaction

In each of these cases the flype is illustrated with respect to a crossing and a tangle R that is a subtangle of the link $K = N(T)$. Cases (i) and

(ii) are taken care of by the trick of the complementary flype. Namely, as in Figure 25, we transfer the crossing of the flype around S^2. Using this crossing we do a tangle flype of the complementary tangle, then we do a horizontal pancake flip and finally an S^2-isotopy, to end up with the right-hand sides of Figure 27.

In case (iii) we note that after the flype the position of the cut points is outside the region of a flyping move that can be performed on the diagram K' to return to the original diagram K, see Figure 28. This means that after performing the return flype the tangle T' is isotopic to the tangle T''. One can now observe that if the original cut produces a rational tangle, then the cut after the returned flype also produces a rational tangle, and this is arithmetically equivalent to the tangle T. More precisely, the tangle T'' is the result of a special cut on $N(T)$.

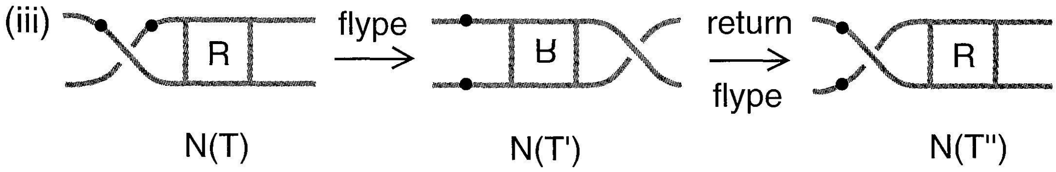


FIGURE 28
Flype and special cut

With the above argument we conclude the proof of the main direction of Theorem 2. From our analysis it follows that:

If $K = N(T)$ is a rational knot diagram with T a rational tangle then, up to bottom twists, any other rational tangle that closes to this knot is available as a cut on the given diagram.

We will now show the converse. We wish to show that if two rational tangles are arithmetically equivalent, then their numerators are isotopic knots. Let T_1, T_2 be rational tangles with $F(T_1) = \frac{p}{q}$ and $F(T_2) = \frac{p}{q'}$, with $|p| > |q|$ and $|p| > |q'|$, and assume first $qq' \equiv 1 \mod p$. If $\frac{p}{q} = [a_1, a_2, \ldots, a_n]$, with n odd, and $\frac{p}{q''} = [a_n, a_{n-1}, \ldots, a_1]$ is the corresponding palindrome continued fraction, then it follows from the Palindrome Theorem that $qq' \equiv 1 \mod p$. Furthermore, it follows by induction that in a product of the form

$$M(a_1)M(a_2)\cdots M(a_n) = \begin{pmatrix} p & q'' \\ q & u \end{pmatrix}$$

we have that $p > q$ and $p > q''$, $q \geq u$ and $q'' \geq u$ whenever $a_1, a_2, \ldots, a_n$ are positive integers. (With the exception in the case of

$M(1)$ where the first two inequalities are replaced by equalities.) The induction step involves multiplying a matrix in the above form by one more matrix $M(a)$, and observing that the inequalities persist in the product matrix.

Hence, in our discussion we can conclude that $|p| > |q''|$. Since $|p| > |q'|$ and $|p| > |q''|$, it follows that $q' = q''$, since they are both reduced residue solutions of a mod p equation with a unique solution. Hence $[a_n, a_{n-1}, \ldots, a_1] = \frac{p}{q'}$, and, by the uniqueness of the canonical form for rational tangles, T_2 has to be:

$$T_2 = [[a_n], [a_{n-1}], \ldots, [a_1]].$$

For these tangles we know that $N(T_1) = N(T_2)$. Let now T_3 be another rational tangle with fraction

$$\frac{p}{q' + kp} = \frac{1}{\frac{q'}{p} + k}.$$

By the Conway Theorem, this is the fraction of the rational tangle

$$\frac{1}{\frac{1}{T_2} + [k]} = T_2 * \frac{1}{[k]}.$$

Hence we have (recall the analysis of the bottom twists):

$$N(\frac{1}{\frac{1}{T_2} + [k]}) \sim N(T_2).$$

Finally, let $F(S_1) = \frac{p}{q}$ and $F(S_2) = \frac{p}{q+kp}$. Then

$$\frac{p}{q + kp} = \frac{1}{\frac{q}{p} + k},$$

which is the fraction of the rational tangle

$$\frac{1}{\frac{1}{S_1} + [k]} = S_1 * \frac{1}{[k]}.$$

Thus

$$N(S_1) \sim N(S_2).$$

The proof of Theorem 2 is now complete. $\square$

We close the section with two remarks.

REMARK 5. In the above discussion about flypes we used the fact that the tangles and flyping tangles involved were rational. One can consider the question of *arbitrary alternating tangles T that close to form links isotopic to a given alternating diagram K*. Our analysis of cuts occurring before and after a flype goes through to show that *for every alternating tangle T, that closes to a diagram isotopic to a given alternating diagram K, there is a cut on the diagram K that produces a tangle that is arithmetically equivalent to T*. Thus it makes sense to consider the collection of tangles that close to an arbitrary alternating link up to this arithmetic equivalence. In the general case of alternating links this shows that on a given diagram of the alternating link we can consider all cuts that produce alternating tangles and thereby obtain all such tangles, up to a certain arithmetical equivalence, that close to links isotopic to K.

Even for rational links there can be more than one equivalence class of such tangles. For example, $N(1/[3] + 1/[3]) = N([-6])$ and $F(1/[3] + 1/[3]) = 2/3$ while $F([-6]) = -6$. Since these fractions have different numerators their tangles (one of which is not rational) lie in different equivalence classes. These remarks lead us to consider the set of arithmetical equivalence classes of altenating tangles that close to a given alternating link and to search for an analogue of Schubert's Theorem in this general setting.

REMARK 6. DNA supercoils, replicates and recombines with the help of certain enzymes. *Site-specific recombination* is one of the ways nature alters the genetic code of an organism, either by moving a block of DNA to another position on the molecule or by integrating a block of alien DNA into a host genome. In [7] it was made possible for the first time to see knotted DNA in an electron micrograph with sufficient resolution to actually identify the topological type of these knots and links. It was possible to design an experiment involving successive DNA recombinations and to examine the topology of the products. In [7] the knotted DNA produced by such successive recombinations was consistent with the hypothesis that all recombinations were of the type of a positive half twist as in $[+1]$. Then D.W. Sumners and C. Ernst [9] proposed a *tangle model for successive DNA recombinations* and showed, in the case of the experiments in question, that there was no other topological possibility for the recombination mechanism than the positive half twist $[+1]$. Their work depends essentially on the classification theorem for rational knots. This constitutes a unique use of topological mathematics as a theoretical underpinning for a problem in molecular biology.

4. Rational knots and their mirror images

In this section we give an application of Theorem 2. An unoriented knot or link K is said to be *achiral* if it is topologically equivalent to its mirror image $-K$. If a link is not equivalent to its mirror image then it is said be *chiral*. One then can speak of the *chirality* of a given knot or link, meaning whether it is chiral or achiral. Chirality plays an important role in the applications of knot theory to chemistry and molecular biology. In [8] the authors find an explicit formula for the number of achiral rational knots among all rational knots with n crossings. It is interesting to use the classification of rational knots and links to determine their chirality. Indeed, we have the following well-known result (for example see [35] and [16], p. 24, Exercise 2.1.4; compare also with [31]):

THEOREM 5. *Let $K = N(T)$ be an unoriented rational knot or link, presented as the numerator of a rational tangle T. Suppose that $F(T) = p/q$ with p and q relatively prime. Then K is achiral if and only if $q^2 \equiv -1 \mod p$. It follows that the tangle T has to be of the form $[[a_1],[a_2],\ldots,[a_k],[a_k],\ldots,[a_2],[a_1]]$ for any integers $a_1,\ldots,a_k$.*

Note that in this description we are using a representation of the tangle with an even number of terms. The leftmost twists $[a_1]$ are horizontal, thus $|p| > |q|$. The rightmost starting twists are then vertical.

Proof. With $-T$ the mirror image of the tangle T, we have that $-K = N(-T)$ and $F(-T) = p/(-q)$. If K is isotopic to $-K$, it follows from the classification theorem for rational knots that either $q(-q) \equiv 1 \mod p$ or $q \equiv -q \mod p$. Without loss of generality we can assume that $0 < q < p$. Hence $2q$ is not divisible by p and therefore it is not the case that $q \equiv -q \mod p$. Hence $q^2 \equiv -1 \mod p$.

Conversely, if $q^2 \equiv -1 \mod p$, then it follows from the Palindrome Theorem that *the continued fraction expansion of p/q has to be palindromic with an even number of terms.* To see this, let $p/q = [c_1,\ldots,c_n]$ with n even, and let $p'/q' = [c_n,\ldots,c_1]$. The Palindrome theorem tells us that $p' = p$ and that $qq' \equiv -1 \mod p$. Thus we have that both q and q' satisfy