2. Lecture 2

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From this we deduce

THEOREM 1.14 ([EG2, BEG, FeV], conjectured in [FV]). *The ring Q_m of m-quasi-invariants is Gorenstein.*

Proof. By Stanley's theorem (see [Eis]), a positively graded Cohen-Macaulay domain A is Gorenstein iff its Poincaré series is a rational function $h(t)$ satisfying the equation $h(t^{-1}) = (-1)^n t^l h(t)$, where l is an integer and n is the dimension of the spectrum of A. Thus the result follows immediately from Proposition 1.13. □

1.6 THE RING OF DIFFERENTIAL OPERATORS ON X_m

Finally, let us introduce the ring $\mathcal{D}(X_m)$ of differential operators on X_m, that is the ring of differential operators with coefficients in $\mathbf{C}(\mathfrak{h})$ mapping Q_m to Q_m. It is clear that this definition coincides with Grothendieck's well-known definition ([Bj]).

THEOREM 1.15 ([BEG]). *$\mathcal{D}(X_m)$ is a simple algebra.*

REMARK 1.16. a) The ring of differential operators on a smooth affine algebraic variety is always simple (see [Bj], Chapter 3).

b) By a result of M. van den Bergh [VdB], for a non-smooth variety, the simplicity of the ring of differential operators implies the Cohen-Macaulay property of this variety.

2. LECTURE 2

We will now see how the ring Q_m appears in the theory of completely integrable systems.

2.1 HAMILTONIAN MECHANICS AND INTEGRABLE SYSTEMS

Recall the basic setup of Hamiltonian mechanics [Ar]. Consider a mechanical system with configuration space X (a smooth manifold). Then the phase space of this system is T^*X, the cotangent bundle on X. The space T^*X is naturally a symplectic manifold, and in particular we have an operation of Poisson bracket on functions on T^*X. A point of T^*X is a pair (x,p), where $x \in X$ is the position and $p \in T_x^*X$ is the momentum. Such pairs are called states of the system. The dynamics of the system $x = x(t)$, $p = p(t)$ depends on the Hamiltonian, or energy function, $E(x,p)$ on T^*X. Given E and the initial state $x(0)$, $p(0)$, one can recover the dynamics $x = x(t)$, $p = p(t)$ from Hamilton's differential equations $\frac{df(x,p)}{dt} = \{f, E\}$. If X is locally identified with $\mathbf{R}^n$ by choosing coordinates $x_1, \dots, x_n$, then T^*X is locally identified with $\mathbf{R}^{2n}$ with coordinates $x_1, \dots, x_n, p_1, \dots, p_n$. In these coordinates, Hamilton's equations may be written in their standard form

$$\dot{x}_i = \frac{\partial E}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial E}{\partial x_i}.$$

A function $I(x,p)$ is called an integral of motion for our system if $\{I, E\} = 0$. Integrals of motion are useful, since for any such integral I the function $I(x(t), p(t))$ is constant, which allows one to reduce the number of variables by 2. Thus, if we are given n functionally independent integrals of motion $I_1, \dots, I_n$ with $\{I_l, I_k\} = 0$ for all $1 \leq l, k \leq n$, then all $2n$ variables x_i, p_i can be excluded, and the system can be completely solved by quadratures. Such a situation is called complete (or Liouville) integrability.

2.2 THE CLASSICAL CALOGERO-MOSER SYSTEM

Quasi-invariants are related to many-particle systems. Consider a system of n particles on the real line $\mathbf{R}$. A potential is an even function

$$U(x) = U(-x), \quad x \in \mathbf{R}.$$

Two particles at points a, b have energy of interaction $U(a - b)$. The total energy of our system of particles is

$$E = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i<j} U(x_i - x_j).$$

Here, x_i are the coordinates of the particles, p_i their momenta. The dynamics of the particles $x_i = x_i(t)$, $p_i = p_i(t)$ is governed by the Hamilton equations with energy function E.

This is a system of nonlinear differential equations, which in general can be difficult to solve explicitly. However, for special potentials this system might be completely integrable. For instance, we will see that this is the case for the Calogero-Moser potential,

$$U(x) = \frac{\gamma}{x^2},$$

γ being a constant.

The Calogero-Moser system has a generalization to arbitrary Coxeter groups. Namely, consider a finite group W generated by reflections acting on the space $\mathfrak{h}$, and keep the notation of the previous section. Fix a W-invariant nondegenerate scalar product $(-,-)$ on $\mathfrak{h}$. It determines a scalar product on $\mathfrak{h}^*$. Define the "energy function"

$$E(x,p) = \frac{(p,p)}{2} + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s(x)^2}, \quad x \in \mathfrak{h}, \quad p \in \mathfrak{h}^*$$

on $T^*\mathfrak{h} = \mathfrak{h} \times \mathfrak{h}^*$, where $\gamma\colon \Sigma \to \mathbf{C}$ is a W-invariant function. Notice that although α_s is defined up to a non zero constant, by homogeneity, E is independent of the choice of α_s. We will call the system defined by E the Calogero-Moser system for W.

If W is the symmetric group S_n, $\mathfrak{h} = \mathbf{C}^n$, then Σ is the set of transpositions $s_{i,j}$, $i < j$, and we can take $\alpha_s = e_i - e_j$, Then we clearly obtain the usual Calogero-Moser system.

Below we will see that the Calogero-Moser system for W is completely integrable.

2.3 The quantum Calogero-Moser system

Let us now discuss quantization of the Calogero-Moser system. We start by quantizing the energy E by formally making the substitution

$$p_j \Rightarrow -i\hbar\frac{\partial}{\partial x_j},$$

where $\hbar$ is a parameter (Planck's constant). This yields the Schrödinger operator

$$\widehat{E} := -\frac{\hbar^2}{2}\Delta + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s^2},$$

where Δ denotes the Laplacian.

In particular, in the case of $W = S_n$ we have

$$\widehat{E} = -\frac{\hbar^2}{2}\Delta + \sum_{i<j} \frac{c}{(x_i - x_j)^2},$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Setting $\beta_s = \frac{\gamma_s}{2\hbar^2}$, we will from now on consider the operator

$$H := -\frac{2}{\hbar^2}\widehat{E} = \Delta - \sum_{s\in\Sigma} \frac{\beta_s(\alpha_s,\alpha_s)}{\alpha_s^2(x)},$$

called the Calogero-Moser operator.

We want to study the stationary Schrödinger equation:

$$H\psi = \lambda\psi\,, \quad \lambda \in \mathbf{C}\,. \tag{3}$$

As in the classical case, it is difficult to say anything explicit about solutions of this equation for a general Schrödinger operator H, but for the Calogero-Moser operator the situation is much better.

DEFINITION 2.1. A *quantum integral* of H is a differential operator M such that

$$[M,H] = 0\,.$$

We are going to show that there are many quantum integrals of H, namely that there are n commuting algebraically independent quantum integrals $M_1,\dots,M_n$ of H. By definition, this means that the quantum Calogero-Moser system is completely integrable.

Once we have found $M_1,\dots,M_n$, observe that for fixed constants $\mu_1,\dots,\mu_n$, the space of solutions of the system

$$\begin{cases} M_1\psi = \mu_1\psi \\ \quad\cdots\cdots \\ M_n\psi = \mu_n\psi \end{cases}$$

is clearly stable under H. We will see that this space is in fact finite dimensional. Therefore, the operators M_i allow one to reduce the problem of solving the partial differential equation $H\psi = \lambda\psi$ to that of solving a system of ordinary linear differential equations. This phenomenon is called quantum complete integrability.

2.4 THE ALGEBRA OF DIFFERENTIAL-REFLECTION OPERATORS

We are now going to explain how to find quantum integrals for H, using the Dunkl-Cherednik method.

First let us fix some notation. Given a smooth affine variety X, we will denote by $\mathcal{D}(X)$ the ring of differential operators on X. We are going to consider the case in which X is the open set U in $\mathfrak{h}$ which is the complement of the divisor of the equation $\delta(x) := \prod_{s\in\Sigma}\alpha_s(x)$. Clearly $\mathcal{D}(U) = \mathcal{D}(\mathfrak{h})[1/\delta(x)]$.

LEMMA 2.2. *An element of $\mathcal{D}(U)$ is completely determined by its action on $\mathbf{C}[U]^W = \mathbf{C}[U/W]$.*

Proof. Recall that the quotient map $\pi\colon U \to U/W$ is finite and unramified. This implies that

$$\mathcal{D}(U) = \mathbf{C}[U] \otimes_{\mathbf{C}[U/W]} \mathcal{D}(U/W).$$

From this we obtain that if $P \in \mathcal{D}(U)$ is such that $Pf = 0$ for all $f \in \mathbf{C}[U/W]$, then $P = 0$. $\square$

We also have the operators on $\mathbf{C}[U]$ given by the action of W. We will denote by $\mathcal{A}$ the algebra of operators on U generated by $\mathcal{D}(U)$ and W, and call it the algebra of differential-reflection operators. The action of W on U induces an action on $\mathcal{D}(U)$, so that the subalgebra $\mathcal{D}(U) \subset \mathcal{A}$ is preserved by conjugation by elements of W. We have:

PROPOSITION 2.3. *$\mathcal{A} = \mathcal{D}(U) \rtimes W$, i.e. every element in $A \in \mathcal{A}$ can be uniquely written as a linear combination*

$$A = \sum_{w \in W} P_w w$$

with $P_w \in \mathcal{D}(U)$.

Proof. The fact that every element in $\mathcal{A}$ can be expressed as a linear combination $\sum_{w\in W} P_w w$ is clear. To show that such an expression is unique, assume $\sum_{w\in W} P_w w = 0$. Take $f \in \mathbf{C}[U]$ such that ${}^w f \neq {}^u f$ for all $w \neq u$ in W, and multiply the operator $\sum P_w w$ on the right by the operator of multiplication by the function f^i, $i \geq 0$. Then we get

$$\sum_{w\in W} P_w \circ ({}^w f)^i w = \sum_{w \in W} P_w w \circ f^i = 0.$$

Applying both sides of this equation to a function $g \in \mathbf{C}[U/W]$ we have

$$\sum_{w\in W} (P_w \circ {}^w f^i)\, g = 0.$$

Thus by Lemma 2.2, $\sum_{w\in W} P_w \circ {}^w f^i = 0$ for all i. Therefore, by Vandermonde's determinant formula, $P_w \circ \prod_{w \neq u} ({}^w f - {}^u f) = 0$ and hence $P_w = 0$, for all $w \in W$, as desired. $\square$

Take $A \in \mathcal{A}$ and write

$$A = \sum_{w \in W} P_w w.$$

We set $m(A) = \sum_{w\in W} P_w \in \mathcal{D}(U)$. Notice that if f is a W-invariant function, then clearly $A(f) = m(A)(f)$ and that, by what we have seen in Lemma 2.2, $m(A)$ is completely determined by its action on invariant functions.

In general, m is not a homomorphism. However:

PROPOSITION 2.4. *Let $\mathcal{A}^W \subset \mathcal{A}$ denote the subalgebra of elements invariant under conjugation by W. Then the restriction of m to $\mathcal{A}^W$ is an algebra homomorphism.*

Proof. If $A \in \mathcal{A}^W$, then clearly $m(A)$ is W-invariant. Now if we take $A, B \in \mathcal{A}^W$ and f a W-invariant function we have that $B(f)$ is also W-invariant. So

$$m(AB)(f) = (AB)(f) = A(B(f)) = A(m(B)(f)) = m(A)(m(B)(f)).$$

Thus $m(AB)$ and $m(A)m(B)$ coincide on W-invariant functions and hence coincide. □

2.5 DUNKL OPERATORS AND SYMMETRIC QUANTUM INTEGRALS

In this subsection we will construct quantum integrals of the Calogero-Moser operator. This construction is due to Heckman [He] and is based on the Dunkl operators, introduced in [Du].

Fix a W-invariant function $c\colon \Sigma \to \mathbf{C}$ such that $\beta_s = c_s(c_s+1)$ for each $s \in \Sigma$. Set $\delta_c := \prod_{s\in\Sigma} \alpha_s(x)^{c_s}$ and define

$$L = \delta_c(x) H \delta_c(x)^{-1}.$$

Then an easy computation shows that

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s},$$

where, for a vector $y \in \mathfrak{h}$, the symbol ∂_y denotes, as usual, the partial derivative in the y direction (notice that using the scalar product we are viewing α_s as a vector in $\mathfrak{h}$ orthogonal to the hyperplane fixed by s).

From now on we will work with L instead of H and study the eigenvalue problem

$$L\psi = \lambda\psi. \tag{4}$$

It is clear that ψ is a solution of this equation if and only if $\delta_c(x)^{-1}\psi$ is a solution of (3).

Since for any $s \in \Sigma$ and $f \in \mathbf{C}[\mathfrak{h}]$ we have that $f(sx) - f(x)$ is divisible by $\alpha_s(x)$, the operator

$$\frac{1}{\alpha_s(x)}(s-1) \in \mathcal{A}$$

maps $\mathbf{C}[\mathfrak{h}]$ to itself.

DEFINITION 2.5. Given $y \in \mathfrak{h}$, we define the *Dunkl operator* D_y on $\mathbf{C}[\mathfrak{h}]$ by

$$D_y := \partial_y + \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)}(s-1).$$

We have the following very important theorem.

THEOREM 2.6 ([Du]). *Let* $y, z \in \mathfrak{h}$. *Then*

$$[D_y, D_z] = 0.$$

Proof. See [Du], [Op]. □

PROPOSITION 2.7 (Heckman [He]). *Let* $\{y_1, \dots, y_n\}$ *be an orthonormal basis of* $\mathfrak{h}$. *Then we have*

$$m(\sum_{i=1}^{n} D_{y_i}^2) = L.$$

Proof. Observe that $m(\sum_{i=1}^n D_{y_i}^2) = \sum_{i=1}^n m(D_{y_i}^2)$, so we need to compute $m(D_y^2)$ for $y \in \mathfrak{h}$. We have $m(D_y^2) = m(D_y m(D_y)) = m(D_y \partial_y)$. A simple computation shows that

$$D_y \partial_y = \partial_y^2 + \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)} (\partial_y (s-1) - \frac{2(\alpha_s, y)}{(\alpha_s, \alpha_s)} \partial_{\alpha_s} s).$$

Thus

$$m(D_y^2) = \partial_y^2 - 2 \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)^2}{(\alpha_s, \alpha_s)\alpha_s(x)} \partial_{\alpha_s}.$$

We get

$$m(\sum_{i=1}^{n} D_{y_i}^2) = \sum_i \partial_{y_i}^2 - 2 \sum_{s \in \Sigma} c_s \frac{\sum_{i=1}^n (\alpha_s, y_i)^2}{(\alpha_s, \alpha_s)\alpha_s(x)} \partial_{\alpha_s} = L,$$

since $\sum_{i=1}^n (\alpha_s, y_i)^2 = (\alpha_s, \alpha_s)$. □

We are now ready to give the construction of quantum integrals of L. Consider the symmetric algebra $S\mathfrak{h} = \mathbf{C}[y_1, \dots, y_n]$ which we can identify, using the fact that the Dunkl operators commute, with the polynomial ring $\mathbf{C}[D_{y_1}, \dots, D_{y_n}] \subset \mathcal{A}$. The restriction of m to $S\mathfrak{h}^W$ is an algebra homomorphism into the ring $\mathcal{D}(U)$ (and in fact into $\mathcal{D}(U/W)$). Since $S\mathfrak{h}^W$ is itself a polynomial ring $\mathbf{C}[q_1, \dots, q_n]$, with $q_1, \dots, q_n$ of degree $d_1, \dots, d_n$,

d_i being the degrees of basic W-invariants, we obtain a polynomial ring of commuting differential operators in $\mathcal{D}(U)$. Given $q \in \mathbf{C}[q_1, \dots, q_n]$ we will denote by L_q the corresponding differential operator. We may assume that $q_1 = \sum_{i=1}^n y_1^2$ so that $L = L_{q_1}$. Thus for every $q \in \mathbf{C}[q_1, \dots, q_n]$, L_q is a quantum integral of the quantum Calogero-Moser system. In particular, the operators $L_{q_1}, \dots, L_{q_n}$ are n algebraically independent pairwise commuting quantum integrals.

Now the eigenvalue problem (4) may be replaced by

$$L_p\psi = \lambda_p\psi$$

for $p \in \mathbf{C}[q_1, \dots, q_n]$, where the assignment $p \to \lambda_p$ is an algebra homomorphism $\mathbf{C}[q_1, \dots, q_n] \to \mathbf{C}$.

In other words, we may say that since $\mathbf{C}[q_1, \dots, q_n] = \mathbf{C}[\mathfrak{h}^*/W] = \mathbf{C}[\mathfrak{h}/W]$, for every point $k \in \mathfrak{h}/W$, we have the eigenvalue problem

$$L_p\psi = p(k)\psi\,. \tag{5}$$

PROPOSITION 2.8. *Near a generic point $x_0 \in \mathfrak{h}$, the system $L_p\psi = p(k)\psi$ has a space of solutions of dimension $|W|$.*

Proof. The proposition follows easily from the fact that the symbols of L_{q_i} are $q_i(\partial)$, and that $\mathbf{C}[y_1, \dots, y_n]$ is a free module over $\mathbf{C}[q_1, \dots, q_n]$ of rank $|W|$. $\square$

2.6 ADDITIONAL INTEGRALS FOR INTEGER VALUED c

If $c_s \notin \mathbf{Z}$, the analysis of the solutions of the equations $L_p\psi = p(k)\psi$ is rather difficult (see [HO]). However, in the case $c \colon \Sigma \to \mathbf{Z}$, the system can be simplified. Let us consider this case. First remark that, since $\beta_s = c_s(c_s+1)$, by changing c_s to $-1-c_s$ if necessary, we may assume that c is non-negative. So we will assume that c takes non-negative integral values and we will denote it by m.

System (5) can be further simplified, if we can find a differential operator M (not a polynomial of $L_{q_1}, \dots, L_{q_n}$) such that $[M, L_p] = 0$ for all $p \in \mathbf{C}[q_1, \dots, q_n]$. Then the operator M will act on the space of solutions of (5), hopefully with distinct eigenvalues. So if μ is such an eigenvalue, the system

$$\begin{cases} L_p\psi = p(k)\,\psi \\ M\psi = \mu\,\psi \end{cases}$$

will have a one dimensional space of solutions and we can find the unique up to scaling solution ψ using Euler's formula.

Such an M exists if and only if $c = m$ has integer values. Namely, we will see that one can extend the homomorphism $\mathbf{C}[q_1, \dots, q_n] \to \mathcal{D}(U)$ mapping $q \to L_q$ to the ring of m-quasi-invariants Q_m.

We start by remarking that under some natural homogeneity assumptions, if such an extension exists, it is unique.

PROPOSITION 2.9. 1) *Assume that* $q \in \mathbf{C}[y_1, \dots, y_n]$ *is a homogeneous polynomial of degree* d. *If there exists a differential operator* M_q *with coefficients in* $\mathbf{C}(\mathfrak{h})$, *of the form*

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

such that $[M_q, L] = 0$, *whose homogeneity degree is* $-d$, *then* M_q *is unique.*

2) *Let* $\mathbf{C}[q_1, \dots, q_n] \subseteq B \subseteq \mathbf{C}[y_1, \dots, y_n]$ *be a graded ring. Assume that we have a linear map* $M\colon B \to \mathcal{D}(U)$ *such that, if* $q \in B$ *is homogeneous of degree* d, *then* $[M_q, L] = 0$, M_q *has homogeneity degree* $-d$, *and*

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

Then M *is a ring homomorphism and* $M_q = L_q$ *for all* $q \in \mathbf{C}[q_1, \dots, q_n]$.

Proof. 1) If there exist two different operators M_q and M'_q with these properties, take $M_q - M'_q$. This operator has degree of homogeneity $-d$, but order smaller than d. Therefore, its symbol $S(x,y)$ is not a polynomial. On the other hand, since the symbol of L is $\sum y_i^2$, we get that $[L, M_q - M'_q] = 0$ implies $\{\sum y_i^2, S(x,y)\} = 0$. Write S in the form $K(x,y)/H(x)$ with K is a polynomial, and $H(x)$ a homogeneous polynomial of positive degree t (we assume that $K(x,y)$ and $H(x)$ have no common irreducible factors). Then

$$0 = \{\sum y_i^2, S(x,y)\} = 2\,\frac{\sum_{i=1}^n y_i K_{x_i}(x,y)H(x) - \sum_{i=1}^n y_i H_{x_i}(x)K(x,y)}{H(x)^2}\,.$$

Since $\sum_{i=1}^n x_i H_{x_i}(x) = tH(x)$, we have $\sum_{i=1}^n y_i H_{x_i}(x)K(x,y) \neq 0$. So $H(x)$ must divide this polynomial and, by our assumptions, this implies that it must divide the polynomial $\sum_{i=1}^n y_i H_{x_i}(x)$ whose degree in x is $t-1$. This is a contradiction.

2) Let $q, p \in B$ be two homogeneous elements. Then $M_q M_p$ and M_{pq} both satisfy the same homogeneity assumptions. Hence they are equal by 1).

Finally if $q \in \mathbf{C}[q_1, \dots, q_n]$, both M_q and L_q satisfy the same homogeneity assumptions. Hence they are equal by 1). □

The required extension to the ring of m-quasi-invariants is then provided by the following

THEOREM 2.10 ([CV1, CV2]). *Let* $c = m\colon \Sigma \to \mathbf{Z}_+$. *The following two conditions are equivalent for a homogeneous polynomial* $q \in \mathbf{C}[\mathfrak{h}^*]$ *of degree* d.

1) *There exists a differential operator*

$$L_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

of homogeneity degree $-d$, *such that* $[L_q, L] = 0$.

2) q *is an* m*-quasi-invariant homogeneous of degree* d.

Using this, we can extend system (5) to the system

$$L_p\psi = p(k)\psi\,, \quad p \in Q_m\,, \quad k \in \operatorname{Spec} Q_m = X_m\,. \tag{6}$$

(Recall that, as a set, $X_m = \mathfrak{h}$.) Near a generic point $x_0 \in \mathfrak{h}$, system (6) has a one dimensional space of solutions, thus there exists a unique up to scaling solution $\psi(k,x)$, which can be expressed in elementary functions. This solution is called the *Baker-Akhiezer function*, and has the form

$$\psi(k,x) = P(k,x)\, e^{(k,x)}$$

with $P(k,x)$ a polynomial of the form $\delta(x)\delta(k) + l.o.t.$ and $e^{(k,x)}$ denotes the exponential function computed in the scalar product (k,x). Furthermore, it can be shown that $\psi(k,x) = \psi(x,k)$ (see [CV1, CV2, FV]).

These results motivate the following terminology. The variety X_m is called *the spectral variety* of the Calogero-Moser system for the multiplicity function m, and Q_m is called *the spectral ring* of this system.

2.7 AN EXAMPLE

EXAMPLE 2.11. Let $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, $m = 1$. As we have seen, Q_m has a basis given by the monomials $\{x^{2i}\} \cup \{x^{2i+3}\}$, $i \geq 0$. Let us set for such a monomial, $L_{x^r} = L_r$, and $\partial = \frac{d}{dx}$. Then we have

$$L_0 = 1, \quad L_2 = \partial^2 - \frac{2}{x}\partial, \quad L_3 = \partial^3 - \frac{3}{x}\partial^2 + \frac{3}{x^2}\partial\,.$$

As for the others, $L_{2j} = L_2^j$, $L_{2j+3} = L_2^j L_3$. (Note that L_1 is not defined). The system (6) in this case is

$$\begin{cases} \psi'' - \dfrac{2}{x}\psi' = k^2\psi\,, \\[2mm] \psi''' - \dfrac{3}{x}\psi'' + \dfrac{3}{x^2}\psi' = k^3\psi\,. \end{cases}$$

The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \dfrac{2}{x}\psi' = k^2\psi\,, \\ \psi'' - (\dfrac{1}{x} + k^2x)\psi' = -k^3x\psi\,. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2x}{kx-1}\psi\,,$$

whose solution (up to constants) is given by $\psi = (kx-1)\,e^{kx}$.

In fact, one can easily calculate ψ_m for a general m.

PROPOSITION 2.12. $\psi_m(k,x) = (x\partial - 2m+1)(x\partial - 2m-1)\cdots(x\partial - 1)\,e^{kx}$.

Proof. We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$(\partial^2 - \frac{2m}{x}\partial)(x\partial - 2m+1) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)$$

as it is easy to verify directly. So using induction on m starting with $m=0$, we get

$$(\partial^2 - \frac{2m}{x}\partial)\psi_m(k,x) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)\psi_{m-1}(k,x) = k^2\psi_m(k,x)\,,$$

and $\psi_m(k,x)$ is our solution. $\square$

3. LECTURE 3

3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function $\psi(k,x)$ for the operator

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)}\partial_{\alpha_s}\,.$$

The way to construct $\psi(k,x)$ is via the Opdam shift operator. Given a function $m\colon \Sigma \to \mathbf{Z}_+$, Opdam showed in [Op1] that there exists a unique W-invariant