2.4 The algebra of differential-reflection operators

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We want to study the stationary Schrödinger equation:

$$H\psi = \lambda\psi, \quad \lambda \in \mathbf{C}. \tag{3}$$

As in the classical case, it is difficult to say anything explicit about solutions of this equation for a general Schrödinger operator H, but for the Calogero-Moser operator the situation is much better.

DEFINITION 2.1. A *quantum integral* of H is a differential operator M such that

$$[M, H] = 0.$$

We are going to show that there are many quantum integrals of H, namely that there are n commuting algebraically independent quantum integrals $M_1, \ldots, M_n$ of H. By definition, this means that the quantum Calogero-Moser system is completely integrable.

Once we have found $M_1, \ldots, M_n$, observe that for fixed constants $\mu_1, \ldots, \mu_n$, the space of solutions of the system

$$\begin{cases} M_1\psi = \mu_1\psi \\ \quad \cdots\cdots \\ M_n\psi = \mu_n\psi \end{cases}$$

is clearly stable under H. We will see that this space is in fact finite dimensional. Therefore, the operators M_i allow one to reduce the problem of solving the partial differential equation $H\psi = \lambda\psi$ to that of solving a system of ordinary linear differential equations. This phenomenon is called quantum complete integrability.

2.4 THE ALGEBRA OF DIFFERENTIAL-REFLECTION OPERATORS

We are now going to explain how to find quantum integrals for H, using the Dunkl-Cherednik method.

First let us fix some notation. Given a smooth affine variety X, we will denote by $\mathcal{D}(X)$ the ring of differential operators on X. We are going to consider the case in which X is the open set U in $\mathfrak{h}$ which is the complement of the divisor of the equation $\delta(x) := \prod_{s\in\Sigma} \alpha_s(x)$. Clearly $\mathcal{D}(U) = \mathcal{D}(\mathfrak{h})[1/\delta(x)]$.

LEMMA 2.2. *An element of $\mathcal{D}(U)$ is completely determined by its action on $\mathbf{C}[U]^W = \mathbf{C}[U/W]$.*

Proof. Recall that the quotient map $\pi\colon U \to U/W$ is finite and unramified. This implies that
$$\mathcal{D}(U) = \mathbf{C}[U] \otimes_{\mathbf{C}[U/W]} \mathcal{D}(U/W)\,.$$
From this we obtain that if $P \in \mathcal{D}(U)$ is such that $Pf = 0$ for all $f \in \mathbf{C}[U/W]$, then $P = 0$. $\square$

We also have the operators on $\mathbf{C}[U]$ given by the action of W. We will denote by $\mathcal{A}$ the algebra of operators on U generated by $\mathcal{D}(U)$ and W, and call it the algebra of differential-reflection operators. The action of W on U induces an action on $\mathcal{D}(U)$, so that the subalgebra $\mathcal{D}(U) \subset \mathcal{A}$ is preserved by conjugation by elements of W. We have:

PROPOSITION 2.3. *$\mathcal{A} = \mathcal{D}(U) \rtimes W$, i.e. every element in $A \in \mathcal{A}$ can be uniquely written as a linear combination*
$$A = \sum_{w \in W} P_w w$$
with $P_w \in \mathcal{D}(U)$.

Proof. The fact that every element in $\mathcal{A}$ can be expressed as a linear combination $\sum_{w\in W} P_w w$ is clear. To show that such an expression is unique, assume $\sum_{w\in W} P_w w = 0$. Take $f \in \mathbf{C}[U]$ such that ${}^w\!f \neq {}^u\!f$ for all $w \neq u$ in W, and multiply the operator $\sum P_w w$ on the right by the operator of multiplication by the function f^i, $i \geq 0$. Then we get
$$\sum_{w\in W} P_w \circ ({}^w\!f)^i w = \sum_{w\in W} P_w w \circ f^i = 0\,.$$
Applying both sides of this equation to a function $g \in \mathbf{C}[U/W]$ we have
$$\sum_{w\in W} (P_w \circ {}^w\!f^i)\, g = 0\,.$$
Thus by Lemma 2.2, $\sum_{w\in W} P_w \circ {}^w\!f^i = 0$ for all i. Therefore, by Vandermonde's determinant formula, $P_w \circ \prod_{w\neq u}({}^w\!f - {}^u\!f) = 0$ and hence $P_w = 0$, for all $w \in W$, as desired. $\square$

Take $A \in \mathcal{A}$ and write
$$A = \sum_{w\in W} P_w w\,.$$
We set $m(A) = \sum_{w\in W} P_w \in \mathcal{D}(U)$. Notice that if f is a W-invariant function, then clearly $A(f) = m(A)(f)$ and that, by what we have seen in Lemma 2.2, $m(A)$ is completely determined by its action on invariant functions.

In general, m is not a homomorphism. However:

PROPOSITION 2.4. *Let $\mathcal{A}^W \subset \mathcal{A}$ denote the subalgebra of elements invariant under conjugation by W. Then the restriction of m to $\mathcal{A}^W$ is an algebra homomorphism.*

Proof. If $A \in \mathcal{A}^W$, then clearly $m(A)$ is W-invariant. Now if we take $A, B \in \mathcal{A}^W$ and f a W-invariant function we have that $B(f)$ is also W-invariant. So

$$m(AB)(f) = (AB)(f) = A(B(f)) = A(m(B)(f)) = m(A)(m(B)(f)).$$

Thus $m(AB)$ and $m(A)m(B)$ coincide on W-invariant functions and hence coincide. □

2.5 DUNKL OPERATORS AND SYMMETRIC QUANTUM INTEGRALS

In this subsection we will construct quantum integrals of the Calogero-Moser operator. This construction is due to Heckman [He] and is based on the Dunkl operators, introduced in [Du].

Fix a W-invariant function $c\colon \Sigma \to \mathbf{C}$ such that $\beta_s = c_s(c_s+1)$ for each $s \in \Sigma$. Set $\delta_c := \prod_{s\in\Sigma} \alpha_s(x)^{c_s}$ and define

$$L = \delta_c(x) H \delta_c(x)^{-1}.$$

Then an easy computation shows that

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s},$$

where, for a vector $y \in \mathfrak{h}$, the symbol ∂_y denotes, as usual, the partial derivative in the y direction (notice that using the scalar product we are viewing α_s as a vector in $\mathfrak{h}$ orthogonal to the hyperplane fixed by s).

From now on we will work with L instead of H and study the eigenvalue problem

$$L\psi = \lambda\psi. \tag{4}$$

It is clear that ψ is a solution of this equation if and only if $\delta_c(x)^{-1}\psi$ is a solution of (3).

Since for any $s \in \Sigma$ and $f \in \mathbf{C}[\mathfrak{h}]$ we have that $f(sx) - f(x)$ is divisible by $\alpha_s(x)$, the operator

$$\frac{1}{\alpha_s(x)}(s-1) \in \mathcal{A}$$

maps $\mathbf{C}[\mathfrak{h}]$ to itself.