

2.4 The algebra of differential-reflection operators

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

We want to study the stationary Schrödinger equation:

$$(3) \quad H\psi = \lambda\psi, \quad \lambda \in \mathbf{C}.$$

As in the classical case, it is difficult to say anything explicit about solutions of this equation for a general Schrödinger operator H , but for the Calogero-Moser operator the situation is much better.

DEFINITION 2.1. A *quantum integral* of H is a differential operator M such that

$$[M, H] = 0.$$

We are going to show that there are many quantum integrals of H , namely that there are n commuting algebraically independent quantum integrals M_1, \dots, M_n of H . By definition, this means that the quantum Calogero-Moser system is completely integrable.

Once we have found M_1, \dots, M_n , observe that for fixed constants μ_1, \dots, μ_n , the space of solutions of the system

$$\begin{cases} M_1\psi = \mu_1\psi \\ \dots\dots \\ M_n\psi = \mu_n\psi \end{cases}$$

is clearly stable under H . We will see that this space is in fact finite dimensional. Therefore, the operators M_i allow one to reduce the problem of solving the partial differential equation $H\psi = \lambda\psi$ to that of solving a system of ordinary linear differential equations. This phenomenon is called quantum complete integrability.

2.4 THE ALGEBRA OF DIFFERENTIAL-REFLECTION OPERATORS .

We are now going to explain how to find quantum integrals for H , using the Dunkl-Cherednik method.

First let us fix some notation. Given a smooth affine variety X , we will denote by $\mathcal{D}(X)$ the ring of differential operators on X . We are going to consider the case in which X is the open set U in \mathfrak{h} which is the complement of the divisor of the equation $\delta(x) := \prod_{s \in \Sigma} \alpha_s(x)$. Clearly $\mathcal{D}(U) = \mathcal{D}(\mathfrak{h})[1/\delta(x)]$.

LEMMA 2.2. *An element of $\mathcal{D}(U)$ is completely determined by its action on $\mathbf{C}[U]^W = \mathbf{C}[U/W]$.*

Proof. Recall that the quotient map $\pi: U \rightarrow U/W$ is finite and unramified. This implies that

$$\mathcal{D}(U) = \mathbf{C}[U] \otimes_{\mathbf{C}[U/W]} \mathcal{D}(U/W).$$

From this we obtain that if $P \in \mathcal{D}(U)$ is such that $Pf = 0$ for all $f \in \mathbf{C}[U/W]$, then $P = 0$. \square

We also have the operators on $\mathbf{C}[U]$ given by the action of W . We will denote by \mathcal{A} the algebra of operators on U generated by $\mathcal{D}(U)$ and W , and call it the algebra of differential-reflection operators. The action of W on U induces an action on $\mathcal{D}(U)$, so that the subalgebra $\mathcal{D}(U) \subset \mathcal{A}$ is preserved by conjugation by elements of W . We have:

PROPOSITION 2.3. $\mathcal{A} = \mathcal{D}(U) \rtimes W$, i.e. every element in $A \in \mathcal{A}$ can be uniquely written as a linear combination

$$A = \sum_{w \in W} P_w w$$

with $P_w \in \mathcal{D}(U)$.

Proof. The fact that every element in \mathcal{A} can be expressed as a linear combination $\sum_{w \in W} P_w w$ is clear. To show that such an expression is unique, assume $\sum_{w \in W} P_w w = 0$. Take $f \in \mathbf{C}[U]$ such that ${}^w f \neq {}^u f$ for all $w \neq u$ in W , and multiply the operator $\sum P_w w$ on the right by the operator of multiplication by the function f^i , $i \geq 0$. Then we get

$$\sum_{w \in W} P_w \circ ({}^w f)^i w = \sum_{w \in W} P_w w \circ f^i = 0.$$

Applying both sides of this equation to a function $g \in \mathbf{C}[U/W]$ we have

$$\sum_{w \in W} (P_w \circ {}^w f^i) g = 0.$$

Thus by Lemma 2.2, $\sum_{w \in W} P_w \circ {}^w f^i = 0$ for all i . Therefore, by Vandermonde's determinant formula, $P_w \circ \prod_{w \neq u} ({}^w f - {}^u f) = 0$ and hence $P_w = 0$, for all $w \in W$, as desired. \square

Take $A \in \mathcal{A}$ and write

$$A = \sum_{w \in W} P_w w.$$

We set $m(A) = \sum_{w \in W} P_w \in \mathcal{D}(U)$. Notice that if f is a W -invariant function, then clearly $A(f) = m(A)(f)$ and that, by what we have seen in Lemma 2.2, $m(A)$ is completely determined by its action on invariant functions.

In general, m is not a homomorphism. However:

PROPOSITION 2.4. *Let $\mathcal{A}^W \subset \mathcal{A}$ denote the subalgebra of elements invariant under conjugation by W . Then the restriction of m to \mathcal{A}^W is an algebra homomorphism.*

Proof. If $A \in \mathcal{A}^W$, then clearly $m(A)$ is W -invariant. Now if we take $A, B \in \mathcal{A}^W$ and f a W -invariant function we have that $B(f)$ is also W -invariant. So

$$m(AB)(f) = (AB)(f) = A(B(f)) = A(m(B)(f)) = m(A)(m(B)(f)).$$

Thus $m(AB)$ and $m(A)m(B)$ coincide on W -invariant functions and hence coincide. \square

2.5 DUNKL OPERATORS AND SYMMETRIC QUANTUM INTEGRALS

In this subsection we will construct quantum integrals of the Calogero-Moser operator. This construction is due to Heckman [He] and is based on the Dunkl operators, introduced in [Du].

Fix a W -invariant function $c: \Sigma \rightarrow \mathbf{C}$ such that $\beta_s = c_s(c_s + 1)$ for each $s \in \Sigma$. Set $\delta_c := \prod_{s \in \Sigma} \alpha_s(x)^{c_s}$ and define

$$L = \delta_c(x) H \delta_c(x)^{-1}.$$

Then an easy computation shows that

$$L = \Delta - \sum_{s \in \Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s},$$

where, for a vector $y \in \mathfrak{h}$, the symbol ∂_y denotes, as usual, the partial derivative in the y direction (notice that using the scalar product we are viewing α_s as a vector in \mathfrak{h} orthogonal to the hyperplane fixed by s).

From now on we will work with L instead of H and study the eigenvalue problem

$$(4) \quad L\psi = \lambda\psi.$$

It is clear that ψ is a solution of this equation if and only if $\delta_c(x)^{-1}\psi$ is a solution of (3).

Since for any $s \in \Sigma$ and $f \in \mathbf{C}[\mathfrak{h}]$ we have that $f(sx) - f(x)$ is divisible by $\alpha_s(x)$, the operator

$$\frac{1}{\alpha_s(x)}(s - 1) \in \mathcal{A}$$

maps $\mathbf{C}[\mathfrak{h}]$ to itself.