3.9 The action of the Cherednik algebra to quasi-invariants

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

3.8 The Levasseur-Stafford theorem and its generalization

Let us now recall a result of Levasseur and Stafford:

THEOREM 3.20 ([LS]). *If G is a finite group acting on a finite dimensional vector space V over the complex numbers, then the ring $\mathcal{D}(V)^G$ is generated by the subrings $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$.*

As an example, notice that if we let $\mathbf{Z}/n\mathbf{Z}$ act on the complex line by multiplication by the n^{th} roots of 1, we deduce that the operator $x\frac{d}{dx}$ can be expressed as a non commutative polynomial in the operators x^n and $\frac{d^n}{dx^n}$, a non-obvious fact. We note also that this theorem has a purely "quantum" nature, i.e. the corresponding "classical" statement, saying that the Poisson algebra $\mathbf{C}[V \times V^*]^G$ is generated, as a Poisson algebra, by $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$, is in fact false, already for $V = \mathbf{C}$ and $G = \mathbf{Z}/n\mathbf{Z}$.

One can prove a similar result for the algebra eH_ce. Namely, recall that the algebra eH_ce contains the subalgebras $\mathbf{C}[\mathfrak{h}]^W$, and $\mathbf{C}[\mathfrak{h}^*]^W$.

THEOREM 3.21 ([BEG]). *If c is generic then the two subalgebras $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ generate eH_ce.*

Notice that if $c = 0$, then $eH_0e = \mathcal{D}(\mathfrak{h})^W$, so Theorem 3.21 reduces to the Levasseur-Stafford theorem.

REMARK. It is believed that this result holds without the assumption of generic c. Moreover, it is known to be true for all c if W is a Weyl group not of type E and F, since in this case Wallach proved that the corresponding classical statement for Poisson algebras holds true. Nevertheless, the genericity assumption is needed for the proof, because, similarly to the proof of the Levasseur-Stafford theorem, it is based on the simplicity of H_c.

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We now go back to the study of Q_m. Notice that the algebra eH_me acts on $\mathbf{C}[\mathfrak{h}]^W$, since e gives the W-equivariant projection of $\mathbf{C}[\mathfrak{h}]$ onto $\mathbf{C}[\mathfrak{h}]^W$. It is clear that this action is by differential operators. For instance, the subalgebra $\mathbf{C}[\mathfrak{h}]^W \subset eH_me$ acts by multiplication. Also, an element $q \in \mathbf{C}[\mathfrak{h}^*]^W \subset eH_me$ acts via the operator $q(D_{x_1}, \ldots, D_{x_n})$. By definition this operator coincides with L_q on $\mathbf{C}[\mathfrak{h}]^W$.

The following important theorem shows that this action extends to Q_m.

THEOREM 3.22 ([BEG]). *There exists a unique representation of the algebra eH_me on Q_m in which an element $q \in \mathbf{C}[\mathfrak{h}]^W$ acts by multiplication and an element $q \in \mathbf{C}[\mathfrak{h}^*]^W$ by L_q.*

Proof. Since by Proposition 3.5, L_q preserves Q_m, we get a uniquely defined representation of the subalgebra of eH_me generated by $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ on Q_m. The result now follows from Theorem 3.21. $\square$

3.10 PROOF OF THEOREM 1.8

Finally we can prove Theorem 1.8.

To do this, observe that as an eH_me-module, Q_m is in the category $\mathcal{O}(eH_me)$, and $\mathbf{C}[\mathfrak{h}^*]^W$ acts locally nilpotently in Q_m (by degree arguments). We can now apply Theorem 3.18 and Theorem 3.17 and deduce that Q_m is a direct sum of modules of the form $eM(0,\tau)$. As a $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(0,\tau) = \mathbf{C}[\mathfrak{h}] \otimes \tau$. On the other hand, by Chevalley's theorem, there is an isomorphism $\mathbf{C}[\mathfrak{h}] \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \mathbf{C}[W]$, commuting with the action of W and $\mathbf{C}[\mathfrak{h}]^W$. Thus we get an isomorphisms of $\mathbf{C}[\mathfrak{h}]^W$-modules

$$eM(0,\tau) \simeq (M(0,\tau))^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes (\mathbf{C}[W] \otimes \tau)^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \tau,$$

proving that $eM(0,\tau)$ and hence Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module. $\square$

EXAMPLE 3.23. For $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$, take the polynomials 1, x^{2m+1}. Notice that $L(1) = L(x^{2m+1}) = 0$ while $s(1) = 1$, $s(x^{2m+1}) = -x^{2m+1}$, $s \in \mathbf{Z}/2$ being the element of order two. It follows that Q_m as a eH_me-module is the direct sum of $\mathbf{C}[x^2] \oplus x^{2m+1}\mathbf{C}[x^2]$. These modules are irreducible. Moreover, $\mathbf{C}[x^2] \simeq eM(0,\mathbf{1})$, $x^{2m+1}\mathbf{C}[x^2] \simeq eM(0,\varepsilon)$, ε being the sign representation.

3.11 PROOF OF THEOREM 1.15

Let I be a nonzero two-sided ideal in $\mathcal{D}(X_m)$. First we claim that I nontrivially intersects Q_m. Indeed, otherwise let $K \in I$ be a lowest order nonzero element in I. Since the order of K is positive, there exists $f \in Q_m$ such that $[K,f] \neq 0$. Then $[K,f] \in I$ is of smaller order than K, a contradiction.

Now let $f \in Q_m$ be an element of I. Then $g = \prod_{w \in W} {}^w f \in I$. But g is W-invariant. This shows that the intersection J of I with the subalgebra H_m in $\mathcal{D}(X_m)$ is nonzero. But H_m is simple by Theorem 3.19, so $J = H_m$. Hence, $1 \in J \subset I$, and $I = \mathcal{D}(X_m)$. $\square$