5. Algebraic proof of Atiyah's $L2$-index theorem

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We now turn to the L^2-index of Section 2. It extends to a homomorphism

$$\mathrm{Index}_G\colon K_0(BG) \to \mathbf{R}$$

as follows. Each $x \in K_0(BG)$ is of the form $f_*(y)$ for some $y = [D] \in K_0(M)$, $f\colon M \to BG$, M a closed smooth manifold and D an elliptic operator on M. Let $\widetilde{D}$ be the lifted operator to $\widetilde{M}$, the G-covering space induced by $f\colon M \to BG$. Then put

$$\mathrm{Index}_G(x) := \mathrm{Index}_G(\widetilde{D})\,.$$

One checks that $\mathrm{Index}_G(x)$ is indeed well-defined, either by direct computation, or by identifying it with $\tau(x)$, where τ denotes the composite of the assembly map $K_0(BG) \to K_0(C_r^*G)$ with the natural trace $K_0(C_r^*G) \to \mathbf{R}$ (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

LEMMA 4.2. *For $H < G$ the following diagram commutes:*

$$\begin{tikzcd} K_0(BH) \arrow[r, "\mathrm{Index}_H"] \arrow[d] & \mathbf{R} \arrow[d, equal] \\ K_0(BG) \arrow[r, "\mathrm{Index}_G"] & \mathbf{R}. \end{tikzcd} \qquad \square$$

Atiyah's L^2-Index Theorem 2.1 for a given G can now be expressed as the statement (as already observed in [10])

$$\mathrm{Index}_G = \mathrm{Index}\colon K_0(BG) \to \mathbf{R}\,.$$

5. ALGEBRAIC PROOF OF ATIYAH'S L^2-INDEX THEOREM

Recall that a group A is said to be *acyclic* if $H_*(BA, \mathbf{Z}) = 0$ for $* > 0$. For G a countable group, there exists an embedding $G \to A_G$ into a countable acyclic group A_G. There are many constructions of such a group A_G available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick's forthcoming work [7]. It follows that the suspension ΣBA_G is contractible, and therefore the inclusion $\{e\} \to A_G$

induces an isomorphism

$$K_0(B\{e\}) \xrightarrow{\cong} K_0(BA_G)\,.$$

Our strategy is as follows. We show that the Atiyah L^2-Index Theorem holds in the special case of acyclic groups, and finish the proof combining the above embedding of a group into an acyclic group.

Proof of Theorem 2.1. If a group A is acyclic, the equation $\mathrm{Index}_A = \mathrm{Index}$ follows from the diagram

$$\begin{tikzcd}
K_0(BA) \arrow[r, "\mathrm{Index}_A"] & \mathbf{R} & K_0(BA) \arrow[l, "\mathrm{Index}"'] \\
K_0(B\{e\}) \arrow[u, "\cong"] \arrow[r, "\mathrm{Index}_{\{e\}}", "\cong"'] & \mathbf{Z} \arrow[u] & K_0(B\{e\}) \arrow[u, "\cong"'] \arrow[l, "\mathrm{Index}"', "\cong"]
\end{tikzcd}$$

because $\mathrm{Index}_{\{e\}} = \mathrm{Index}$ on the bottom line. For a general group G, consider an embedding into an acyclic group A_G and complete the proof by using Lemma 3.1, together with Lemmas 4.1 and 4.2.

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