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IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREES FOUR AND FIVE AND RELATED DIOPHANTINE SYSTEMS

by Ajai CHOUDHRY

ABSTRACT. In this paper, we obtain parametric ideal non-symmetric solutions in integers of the Tarry-Escott problem of degrees four and five, that is, of the system of simultaneous equations $\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r$, $r = 1, 2, \ldots, k$ where k is 4 or 5. We use these non-symmetric solutions to obtain parametric solutions of the two diophantine systems $\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r$, $r = 1, 2, \ldots, k, k+2$ where k is 4 or 5.

1. INTRODUCTION

This paper is a sequel to my earlier paper [1] regarding the Tarry-Escott problem. It would be recalled that very little is known about ideal non-symmetric solutions of the Tarry-Escott problem of degree k when $k > 3$. When $k = 4$, the only known parametric ideal non-symmetric solution of the Tarry-Escott problem is given in [1]. This solution is in terms of polynomials of degree 8 in two parameters. When $k = 5$, only a single numerical solution seems to have been published [2, p. 27]. No non-symmetric solutions have been published for $k > 5$.

In this paper, we will obtain parametric ideal non-symmetric solutions of the Tarry-Escott problem of degrees four and five. The parametric solutions of the Tarry-Escott problem of degree four obtained in this paper are more general and much simpler as compared to the parametric solution of this problem given in [1].

It has already been shown in [1] how ideal non-symmetric solutions of the Tarry-Escott problem of degree k may be used to generate solutions of the system of equations

$$\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \quad r = 1, 2, \ldots, k, k+2 \tag{1.1}$$

by applying a theorem of Gloden [2, p. 24]. Applying this procedure to the non-symmetric ideal solutions of degrees four and five obtained in this paper, we get parametric solutions of (1.1) when $k = 4$ or $k = 5$.

2. Ideal non-symmetric solutions of the Tarry-Escott problem of degree four

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree four, we have to obtain a solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4. \tag{2.1}$$

We first observe that the system of equations

$$X_1^r + X_2^r + X_3^r = Y_1^r + Y_2^3 + Y_3^r, \qquad r = 1, 2, 4, \tag{2.2}$$

reduces to

$$X_1^2 + X_1X_2 + X_2^2 = Y_1^2 + Y_1Y_2 + Y_2^2, \tag{2.3}$$

if we take $X_3 = -X_1 - X_2$ and $Y_3 = -Y_1 - Y_2$. A solution of (2.3) in terms of arbitrary parameters m, n, x, y, is given by

$$\begin{aligned} X_1 &= (m + 2n)x + (-m + n)y, \\ X_2 &= (-2m - n)x + (-m - 2n)y, \\ Y_1 &= (m - n)x + (-m - 2n)y, \\ Y_2 &= (-2m - n)x + (-m + n)y, \end{aligned} \tag{2.4}$$

and we now get

$$\begin{aligned} X_3 &= (m - n)x + (2m + n)y, \\ Y_3 &= (m + 2n)x + (2m + n)y. \end{aligned} \tag{2.5}$$

It follows from this solution of the system of equations (2.2) that if we take

$$
(2.6)\qquad
\begin{aligned}
a_1 &= (m_1+2n_1)x_1+(-m_1+n_1)y_1\,,\\
a_2 &= (-2m_1-n_1)x_1+(-m_1-2n_1)y_1\,,\\
a_3 &= (m_1-n_1)x_1+(2m_1+n_1)y_1\,,\\
a_4 &= (m_2+2n_2)x_2+(2m_2+n_2)y_2\,,\\
a_5 &= (-2m_2-n_2)x_2+(-m_2+n_2)y_2\,,\\
a_6 &= (m_2-n_2)x_2+(-m_2-2n_2)y_2\,,\\
b_1 &= (m_2+2n_2)x_2+(-m_2+n_2)y_2\,,\\
b_2 &= (-2m_2-n_2)x_2+(-m_2-2n_2)y_2\,,\\
b_3 &= (m_2-n_2)x_2+(2m_2+n_2)y_2\,,\\
b_4 &= (m_1-n_1)x_1+(-m_1-2n_1)y_1\,,\\
b_5 &= (-2m_1-n_1)x_1+(-m_1+n_1)y_1\,,\\
b_6 &= (m_1+2n_1)x_1+(2m_1+n_1)y_1\,,
\end{aligned}
$$

then

$$(2.7)\qquad \sum_{i=1}^{6} a_i^r = \sum_{i=1}^{6} b_i^r\,,$$

is identically satisfied for $r = 1, 2$ and 4. Therefore, to obtain a solution of (2.1), we only have to choose m_i, n_i, x_i, y_i, such that (2.7) also holds for $r = 3$ and, at the same time, the additional condition $a_6 = b_6$ is satisfied.

When $r = 3$, (2.7) reduces to the equation

$$(2.8)\qquad m_1n_1(m_1+n_1)x_1y_1(x_1+y_1) = m_2n_2(m_2+n_2)x_2y_2(x_2+y_2)$$

which is to be solved together with the additional condition

$$(2.9)\qquad (m_2-n_2)x_2+(-m_2-2n_2)y_2 = (m_1+2n_1)x_1+(2m_1+n_1)y_1\,.$$

To solve the simultaneous equations (2.8) and (2.9), we write

$$(2.10)\qquad m_2 = tm_1\,,\quad n_2 = tn_1\,,\quad x_1 = px_2\,,\quad y_1 = qy_2\,,$$

when (2.8) is readily solved to get

$$(2.11)\qquad x_2 = pq^2 - t^3\,,\quad y_2 = -p^2q + t^3\,.$$

Next, we find x_1, y_1 from (2.10), then solve (2.9) for m_1, n_1 to get

$$(2.12)\qquad m_1 = pq - 2pt + t^2\,,\quad n_1 = pq + pt - 2t^2\,,$$

and then (2.10) gives

$$(2.13)\qquad m_2 = t(pq - 2pt + t^2)\,,\quad n_2 = t(pq + pt - 2t^2)\,.$$

We now substitute the values of m_1, n_1, m_2, n_2, x_1, x_2, y_1, y_2 in (2.6) to get the following non-symmetric solution of the Tarry-Escott problem of degree four:

$$
(2.14)\quad
\begin{aligned}
a_1 &= p^3q^3 - p^3q^2t - p^2qt^3 + pqt^4 + pt^5 - qt^5\,,\\
a_2 &= p^3q^2t - p^2q^2t^2 + p^2qt^3 - p^2t^4 - pq^2t^3 + qt^5\,,\\
a_3 &= -p^3q^3 + p^2q^2t^2 + p^2t^4 + pq^2t^3 - pqt^4 - pt^5\,,\\
a_4 &= -p^3q^2t + p^3qt^2 + p^2q^3t - pq^2t^3 - pt^5 + t^6\,,\\
a_5 &= -p^3qt^2 - p^2q^3t + p^2q^2t^2 + p^2qt^3 + pqt^4 - t^6\,,\\
b_1 &= -p^3qt^2 + p^2q^3t + p^2qt^3 - pq^2t^3 - pqt^4 + pt^5\,,\\
b_2 &= p^3q^2t - p^2q^3t + p^2q^2t^2 - p^2qt^3 - pt^5 + t^6\,,\\
b_3 &= -p^3q^2t + p^3qt^2 - p^2q^2t^2 + pq^2t^3 + pqt^4 - t^6\,,\\
b_4 &= p^3q^3 - p^3q^2t + p^2t^4 - pq^2t^3 - pt^5 + qt^5\,,\\
b_5 &= -p^3q^3 + p^2q^2t^2 + p^2qt^3 - p^2t^4 + pqt^4 - qt^5\,.
\end{aligned}
$$

While this solution is in terms of polynomials of degree six in three parameters, it yields simpler solutions in terms of polynomials of degree three if we consider q and t as constants. For example, taking $q = 1$, $t = -1$, we get the following ideal non-symmetric solution of the Tarry-Escott problem of degree four:

$$
(2.15)\quad
\begin{aligned}
a_1 &= 2p^3 + p^2 + 1\,, & b_1 &= -p^3 - 2p^2 - p\,,\\
a_2 &= -p^3 - 3p^2 + p - 1\,, & b_2 &= -p^3 + 3p^2 + p + 1\,,\\
a_3 &= -p^3 + 2p^2 - p\,, & b_3 &= 2p^3 - p^2 - 1\,,\\
a_4 &= 2p^3 - p^2 + 2p + 1\,, & b_4 &= 2p^3 + p^2 + 2p - 1\,,\\
a_5 &= -p^3 + p^2 + p - 1\,, & b_5 &= -p^3 - p^2 + p + 1\,.
\end{aligned}
$$

In this solution we may take p as a rational parameter. Integer solutions of (2.1) are obtained by multiplying any rational numerical solution by a suitable constant. Substituting $p = -2$ in the above solution, we get, after suitable re-arrangement, the following numerical solution:

$$(-23)^r + (-11)^r + (-7)^r + 9^r + 18^r = (-21)^r + (-17)^r + 2^r + 3^r + 19^r$$

where $r = 1, 2, 3, 4$. Adding the constant 24 to all the terms, we get the following solution in positive integers:

$$1^r + 13^r + 17^r + 33^r + 42^r = 3^r + 7^r + 26^r + 27^r + 43^r\,,$$

where $r = 1, 2, 3, 4$.

We may apply the theorem of Gloden [2, p. 24] to the three-parameter ideal non-symmetric solution obtained above to derive a solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r\,, \qquad r = 1,2,3,4,6\,, \tag{2.16}$$

in terms of polynomials of degree six in three parameters. We, however, restrict ourselves to applying this theorem to the simpler solution (2.15), and obtain the following solution of the system of equations (2.16):

$$\begin{aligned}
a_1 &= 9p^3 + 5p^2 - 3p + 5\,, & b_1 &= -6p^3 - 10p^2 - 8p\,,\\
a_2 &= -6p^3 - 15p^2 + 2p - 5\,, & b_2 &= -6p^3 + 15p^2 + 2p + 5\,,\\
a_3 &= -6p^3 + 10p^2 - 8p\,, & b_3 &= 9p^3 - 5p^2 - 3p - 5\,,\\
a_4 &= 9p^3 - 5p^2 + 7p + 5\,, & b_4 &= 9p^3 + 5p^2 + 7p - 5\,,\\
a_5 &= -6p^3 + 5p^2 + 2p - 5\,, & b_5 &= -6p^3 - 5p^2 + 2p + 5\,.
\end{aligned} \tag{2.17}$$

When $p = -2$, this leads to the following solution of the system of equations (2.16):

$$(-101)^r + (-41)^r + (-21)^r + 59^r + 104^r = (-91)^r + (-71)^r + 24^r + 29^r + 109^r$$

where $r = 1,2,3,4,6$.

We note that additional parametric non-symmetric solutions of the Tarry-Escott problem of degree four may be obtained by taking a_i, b_i, as in (2.6), and instead of imposing the condition $a_6 = b_6$, we reduce one term on either side by solving (2.8) together with another condition such as $a_4 = b_6$ or $a_5 = b_6$. Solutions obtained in this manner are of degrees $6, 7$ or 8 in terms of three parameters.

3. Ideal non-symmetric solutions of the Tarry-Escott problem of degree five

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree five, we have to obtain a solution of the system of equations

$$\sum_{i=1}^{6} a_i^r = \sum_{i=1}^{6} b_i^r\,, \qquad r = 1,2,3,4,5\,. \tag{3.1}$$

We will choose $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ as in (2.6) when (3.1) holds identically for $r = 1, 2, 4$. For $r = 3$, equation (3.1) reduces to (2.8) while for $r = 5$ it reduces to the equation:

$$\begin{aligned}(3.2)\quad & m_1n_1(m_1+n_1)x_1y_1(x_1+y_1)(m_1^2+m_1n_1+n_1^2)(x_1^2+x_1y_1+y_1^2)\\ &= m_2n_2(m_2+n_2)x_2y_2(x_2+y_2)(m_2^2+m_2n_2+n_2^2)(x_2^2+x_2y_2+y_2^2)\,.\end{aligned}$$

It therefore suffices to solve equation (2.8) together with the following equation:

$$(3.3)\quad (m_1^2+m_1n_1+n_1^2)(x_1^2+x_1y_1+y_1^2) = (m_2^2+m_2n_2+n_2^2)(x_2^2+x_2y_2+y_2^2)\,.$$

We take x_1, y_1, m_1, m_2, n_2 such that

$$(3.4)\qquad \begin{aligned} x_1 &= (t^2+t-1)x_2\,,\\ y_1 &= (t+1)^2y_2\,,\\ m_1 &= tx_2+ty_2\,,\\ m_2 &= (t^2+t-1)x_2+(t+1)^2y_2\,,\\ n_2 &= (-t^2-t)n_1\,.\end{aligned}$$

Substituting these values of x_1, y_1, m_1, m_2, n_2 in (2.8) and solving for n_1, we get

$$(3.5)\qquad n_1 = -\frac{(t^4+2t^3+t^2-1)x_2+(t^4+2t^3+t^2+t+1)y_2}{t^3+t^2-t-1}\,,$$

and now (3.4) gives

$$(3.6)\qquad n_2 = \frac{(t^5+2t^4+t^3-t)x_2+(t^5+2t^4+t^3+t^2+t)y_2}{t^2-1}\,.$$

On substituting the values of n_1, n_2 given by (3.5) and (3.6), and the values of x_1, y_1, m_1, m_2 given by (3.4) in equation (3.3), we get the equation:

$$\begin{aligned}&(tx_2+(t+1)y_2)((t^2+t-1)x_2+(t+1)y_2)((t^2+t-1)x_2+(t^2+t)y_2)\\ &\times (t+2)((2t^5+5t^4+3t^3-t^2-t+1)x_2+(t^5-6t^3-8t^2-4t-1)y_2)=0\,.\end{aligned}$$

Equating any of the first four factors on the left-hand side of this equation to zero leads either to trivial solutions or to known symmetric solutions of the Tarry-Escott problem of degree five. However, on equating the last factor to zero, we get

$$(3.7)\qquad \begin{aligned} x_2 &= t^5-6t^3-8t^2-4t-1\,,\\ y_2 &= -(2t^5+5t^4+3t^3-t^2-t+1)\,,\end{aligned}$$

and now, using the relations (3.4), (3.5) and (3.6), we get the values of $x_1, y_1, m_1, m_2, n_1, n_2$. The values of $x_1, y_1, x_2, y_2, m_1, m_2, n_1, n_2$, may now be substituted in (2.6) to get a non-symmetric solution of the Tarry-Escott problem of degree five. After removing the common factors, this solution may be written as follows:

$$\begin{aligned}
a_1 &= -3t^{11} - 20t^{10} - 58t^9 - 94t^8 - 106t^7 - 100t^6 - 40t^5 \\
&\qquad + 50t^4 + 50t^3 + 2t^2 - 5t\,, \\
a_2 &= 3t^{11} + 16t^{10} + 38t^9 + 71t^8 + 128t^7 + 149t^6 + 56t^5 \\
&\qquad - 37t^4 - 22t^3 + 5t^2 + t - 3\,, \\
a_3 &= 4t^{10} + 20t^9 + 23t^8 - 22t^7 - 49t^6 - 16t^5 \\
&\qquad - 13t^4 - 28t^3 - 7t^2 + 4t + 3\,, \\
a_4 &= 8t^{10} + 52t^9 + 127t^8 + 148t^7 + 85t^6 + 22t^5 \\
&\qquad + t^4 - 14t^3 - 23t^2 - 4t + 3\,, \\
a_5 &= 3t^{11} + 14t^{10} + 16t^9 - 29t^8 - 98t^7 - 89t^6 + 4t^5 + 55t^4 \\
&\qquad + 40t^3 + 13t^2 - 7t - 3\,, \\
a_6 &= -3t^{11} - 22t^{10} - 68t^9 - 98t^8 - 50t^7 + 4t^6 - 26t^5 \\
&\qquad - 56t^4 - 26t^3 + 10t^2 + 11t\,, \\
b_1 &= 2t^{10} + 13t^9 + 58t^8 + 151t^7 + 190t^6 + 79t^5 \\
&\qquad - 44t^4 - 41t^3 - 2t^2 - t\,, \\
b_2 &= -3t^{11} - 19t^{10} - 56t^9 - 116t^8 - 164t^7 - 104t^6 + 34t^5 \\
&\qquad + 76t^4 + 28t^3 + 4t^2 - t - 3\,, \\
b_3 &= 3t^{11} + 17t^{10} + 43t^9 + 58t^8 + 13t^7 - 86t^6 - 113t^5 \\
&\qquad - 32t^4 + 13t^3 - 2t^2 + 2t + 3\,, \\
b_4 &= 10t^{10} + 59t^9 + 140t^8 + 167t^7 + 98t^6 - t^5 \\
&\qquad - 58t^4 - 31t^3 + 14t^2 + 7t\,, \\
b_5 &= -3t^{11} - 23t^{10} - 67t^9 - 88t^8 - 25t^7 + 68t^6 + 71t^5 \\
&\qquad + 14t^4 - 13t^3 - 16t^2 - 2t + 3\,, \\
b_6 &= 3t^{11} + 13t^{10} + 8t^9 - 52t^8 - 142t^7 - 166t^6 - 70t^5 \\
&\qquad + 44t^4 + 44t^3 + 2t^2 - 5t - 3\,.
\end{aligned} \tag{3.8}$$

Here t is an arbitrary rational parameter and integer solutions of (3.1) are obtained by multiplying any rational numerical solution by a suitable constant. The a_i, b_i obtained above satisfy the relation $\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} b_i = 0$.

It therefore follows from the theorem of Gloden [2, p. 24] that these a_i, b_i must also satisfy the relation

$$\sum_{i=1}^{6} a_i^7 = \sum_{i=1}^{6} b_i^7 .$$

This is also verified by direct computation. Hence the a_i, b_i given by (3.8) constitute a solution of the following system of equations:

$$\sum_{i=1}^{6} a_i^r = \sum_{i=1}^{6} b_i^r , \qquad r = 1,2,3,4,5,7 .$$

As a numerical example, when $t = -3$, we get, after removal of common factors and suitable re-arrangement, the following solution

$$\begin{aligned}(-19323)^r + (-18689)^r + 3117^r + 5111^r + 14212^r + 15572^r \\ = (-20023)^r + (-17828)^r + 1017^r + 9787^r + 10236^r + 16811^r\end{aligned}$$

where $r = 1,2,3,4,5,7$.

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