Lattices, l2-Betti numbers, deficiency, and knot groups

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LATTICES, ℓ_2-BETTI NUMBERS, DEFICIENCY, AND KNOT GROUPS

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In an introductory lecture to the "Borel-Seminar" on ℓ_2-homology in Berne, Summer 2002, I described among various other classes a new class of finitely presented groups where the first ℓ_2-Betti number vanishes. Namely, all infinite lattices in arbitrary connected Lie groups apart from special exceptions. These exceptions are the lattices commensurable with those in $\mathrm{PSL}_2(\mathbf{R})$, the isometry group of the hyperbolic plane. In the general case it follows that the deficiency of the lattice is ≤ 1. The lattices with vanishing first ℓ_2-Betti number and deficiency equal to 1 are of special interest. They are examined in Part Two of this paper.

After writing down the present detailed survey of these aspects of my lecture I learned that the class was not that new: It had appeared, not long before, in a paper by John Lott [L] on the deficiency of lattices. In spite of a considerable overlap between the two papers there are some differences in methods and motivation; the present text can be considered as a compendium to [L]. As for the methods, for example, my treatment of harmonic L_2-forms is reduced to the cocompact case (thanks to Gaboriau's proportionality principle) and thus does not use the singular ℓ_2-theory of Cheeger-Gromov nor their interesting but complicated approach to the non-cocompact case in [Ch-G2]. Or, in Part Two, the additivity of (virtual) cohomology dimensions for group extensions is a simplifying tool. My motivation for Part Two was the discussion of knot groups, since they have exactly the respective properties; this is not in [L], where the motivation for the special lattices is to show that, apart from few exceptions, lattices have deficiency ≤ 0 (which contains various special results by Lubotzky).

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PART ONE: LATTICES AND FIRST ℓ_2-BETTI NUMBER

1. SURVEY AND PRELIMINARIES

1.1. It is well-known that the Lie group $L = \mathrm{PSL}_2(\mathbf{R})$ contains lattices Γ which are non-Abelian free, and lattices which are isomorphic to the fundamental group of closed surfaces of genus ≥ 2 (we will call them in short "surface groups"). The former are non-cocompact and the latter cocompact (uniform), and these are the only possibilities for a torsion-free lattice in $\mathrm{PSL}_2(\mathbf{R})$. The symmetric space $M = L/K$ where K is a maximal compact subgroup of L, the hyperbolic plane, is the universal covering of a $K(\Gamma, 1)$-manifold of dimension 2, open in the non-cocompact case and closed in the cocompact case. Since the cohomology dimension $\mathrm{cd}(\Gamma)$ [B,Ch.VIII], equal to the geometric dimension except possibly for $\mathrm{cd} = 2$, will play a certain role in this paper we note here that $\mathrm{cd}(\Gamma)$ is 1 in the former and 2 in the latter case. The first ℓ_2-Betti number $\beta_1(\Gamma)$ of these lattices is $\neq 0$.

If we admit torsion then the lattice Γ is virtually torsion-free, i.e. contains a subgroup of finite index which is torsion-free. Thus all the above remarks apply "virtually". The virtual cohomology dimension $\mathrm{vcd}(\Gamma)$ is 1 or 2 respectively; Γ is virtually non-Abelian free or a virtual surface group, and $\beta_1(\Gamma) \neq 0$.

1.2. In Part One we show, in the general case of arbitrary connected Lie groups L, that the non-vanishing of β_1 of a lattice is exceptional, in the following sense. *If for a lattice Γ in a connected Lie group the first ℓ_2-Betti number is non-zero then Γ is commensurable with a torsion-free lattice Δ in* $\mathrm{PSL}_2(\mathbf{R})$ (this means that there is a subgroup Γ_0 of Γ of finite index and an exact sequence

$$1 \longrightarrow N \longrightarrow \Gamma_0 \longrightarrow \Delta \longrightarrow 1$$

where N is finite and Δ a torsion-free lattice in $\mathrm{PSL}_2(\mathbf{R})$, i.e. a surface group or a non-Abelian free group). Thus in general the first ℓ_2-Betti number of a lattice is zero. This implies properties of the deficiency of a lattice, and of the signature of a 4-manifold with fundamental group isomorphic to a lattice in a connected Lie group.

1.3. We first consider (Section 2) the case of a connected semisimple linear Lie group L without compact factors. Here we discuss the non-vanishing of β_k of a lattice Γ for arbitrary $k > 0$. Using deep results (see Section 2) it turns out that $\beta_k(\Gamma) = 0$ for all k except possibly for the middle dimension of the symmetric space $M = L/K$. This has applications. We then turn in Section 3 to the non-vanishing of β_1. Arbitrary Lie groups are considered in Section 4; the general case is reduced in several steps to the case of a semisimple linear Lie group without compact factors.

1.4. DEFINITIONS

a) A lattice Γ in a connected Lie group L is a discrete subgroup of L such that L/Γ has finite volume. It is said cocompact if L/Γ is compact. A lattice can only be finite if L is compact.

b) Lattices are finitely presentable. If a finitely presentable group G has a presentation with α generators and β defining relations the difference $\alpha - \beta$ is called the deficiency of the presentation. The deficiency $\mathrm{def}(G)$ is the maximum of the deficiencies over all presentations.

c) The geometric dimension $\mathrm{gdim}(G)$ of a group G is the minimum of the dimensions of all $K(G,1)$-spaces. In this note "space" will always mean cell-complex. We recall that the cohomology dimension $\mathrm{cd}(G)$ [cf [B]) is equal to $\mathrm{gdim}(G)$ except possibly for $\mathrm{cd}(G) = 2$. In any case $\mathrm{gdim}(G) \leq 2$ implies $\mathrm{cd}(G) \leq 2$ and G is torsion-free.

d) For the definition of ℓ_2-Betti numbers we refer to [E] where further references can be found. Here we just recall that one considers a free cocompact G-space Y, in other words a regular covering of a finite cell-complex X with G as covering transformation group. The reduced ℓ_2-homology group $H_i(Y)$ is a finitely generated Hilbert G-module and its von Neumann dimension relative to G is the ith Betti number $\beta_i(Y, \mathrm{rel}\ G)$. If Y is the universal covering $\widetilde{X}$ of X then $G = \pi_1(X)$ and $\beta_i(\widetilde{X}, \mathrm{rel}\ \pi_1(X))$ is often written $\beta_i(X)$. The combinatorial Euler characteristic $\chi(X)$ of the cell-complex X is equal to the alternating sum of the ℓ_2-Betti numbers of X (exactly as in the classical Euler-Poincaré formula with ordinary Betti numbers). Note that $\beta_0(X)$ and $\beta_1(X)$ only depend on the fundamental group $\pi_1(X) = G$ and can be written $\beta_0(G)$ and $\beta_1(G)$, and that $\beta_0(G) = 0$ if and only if G is infinite. More generally the ℓ_2-Betti numbers of G are defined as those of $K(G,1)$ provided there is a finite $K(G,1)$-complex (or for $i \leq n$ if there is a $K(G,1)$ with finite $(n+1)$-skeleton).

REMARK. ℓ_2-Betti numbers have also been defined for more general free G-spaces (see [Ch-G]) and even for arbitrary G-spaces [L, Ch.6]. In the case of a free cocompact action they agree, of course, with those described above.

In particular one has ℓ_2-Betti numbers for groups G without any finiteness condition.

If, in our context, Γ is non-cocompact we obtain from the symmetric space an open $K(\Gamma, 1)$-manifold. There one has to use the more general type of ℓ_2-Betti numbers in order to obtain information about harmonic L_2-forms (see [Ch-G2]). We avoid that procedure by reducing the approach to the cocompact case.

1.5. PRELIMINARIES. We recall for later use at different places some facts concerning $\beta_i(G)$ of a finitely presented infinite group G.

PROPOSITION 1.1. *If G_1 is a subgroup of finite index in G then, for all $i \geq 0$, $\beta_i(G) \neq 0$ if and only if $\beta_i(G_1) \neq 0$.*

PROPOSITION 1.2. *If N is a finite normal subgroup of G then $\beta_1(G) \neq 0$ if and only if $\beta_1(G/N) \neq 0$.*

Proposition 1.1 follows from the fact that for a subgroup of finite index the β_i are multiplied by the index.

To prove Proposition 1.2 consider the Lyndon-Hochschild-Serre spectral sequence leading from G/N to G. Since $\beta_0(G/N) = \beta_1(N) = 0$ and $\beta_0(N) \neq 0$ the only term contributing to $\beta_1(G)$ is $H_1(G/N; H_0(N))$ (reduced ℓ_2-homology). Thus $H_1(G) \neq 0$ if and only if $H_1(G/N) \neq 0$.

PROPOSITION 1.3 (The Cheeger-Gromov Theorem [Ch-G]). *If the finitely presented group G admits an infinite amenable normal subgroup N then $\beta_1(G) = 0$ (actually, in the more general sense $\beta_i(G) = 0$ for all i).*

PROPOSITION 1.4. *If $\beta_1(G) = 0$ then the deficiency* $\operatorname{def}(G)$ *is ≤ 1, and* $\operatorname{def}(G) = 1$ *implies that* $\operatorname{gdim}(G) = \operatorname{cd}(G) \leq 2$.

To prove this let Z be the finite 2-dimensional cell complex constructed from a presentation with deficiency equal to $\operatorname{def}(G)$, and $\widetilde{Z}$ its universal covering. We recall from [E] that

$$\chi(Z) = 1 - \operatorname{def}(G) = -\beta_1(G) + \beta_2(Z).$$

If $\beta_1(G) = 0$ then $\operatorname{def}(G) \leq 1$, and if $\operatorname{def}(G)$ is equal to 1 then $\beta_2(Z) = 0$ whence $H_2(\widetilde{Z}) = 0$. Since integral finite cycles are contained in $H_2(\widetilde{Z})$ it follows that ordinary integral homology in dimension 2 of $\widetilde{Z}$ is 0; thus $\widetilde{Z}$ is contractible, and Z is a finite $K(G,1)$ of dimension two.

Since surface groups of genus ≥ 2 and non-Abelian free groups have deficiency ≥ 2 it follows from Proposition 1.4 that their first ℓ_2-Betti number is $\neq 0$, and the same holds in the virtual case.

We will also make use of the following classical result (Stallings' Theorem, see [C] where further references are given) concerning cohomology dimension.

THEOREM 1.5. *A finitely generated group of cohomology dimension one is free.*

It has been proved by Swan that the statement is true for *all* groups. We will not need that generalization of Theorem 1.5.

2. ℓ_2-BETTI NUMBERS OF LATTICES IN SEMISIMPLE LINEAR GROUPS

2.1. Let Γ be a torsion-free lattice in the connected semisimple linear Lie group L without compact factors.

The Riemannian symmetric space $M = L/K$, where K is a maximal compact subgroup, is a contractible free Γ-manifold of dimension d. It is the universal covering of a $K(\Gamma,1)$-manifold X, of dimension d, closed or open according to whether Γ is cocompact or not.

We first look at the case where X is compact. By the theorem of Dodziuk [D] the reduced ℓ_2-cohomology space $H^k(M)$ of M is Γ-isomorphic to the space $\mathcal{H}^k(M)$ of harmonic L_2-forms of degree k.

If we now assume that $\beta_k(\Gamma) \neq 0$ then $\mathcal{H}^k(M) \neq 0$ and therefore M admits non-zero harmonic L_2-forms of degree k. According to Borel-Wallach [B-W, Section 5 in Chap. II] and Connes-Moscovici [C-M, Theorem 6.1 ff] there is only one dimension q where such harmonic q-forms $\neq 0$ can exist, namely $q = \frac{1}{2}\dim M = \frac{1}{2}d$. Thus the dimension d of M must be $2k$.

2.2. If the lattice Γ is non-cocompact we use the proportionality principle of Gaboriau [G, Cor. 0.2]. It says that for fixed k the Betti numbers $\beta_k(\Gamma)$ of all lattices Γ in L differ only by a positive factor. Thus if $\beta_k(\Gamma) \neq 0$ the same is true for any cocompact lattice in L; and such lattices always exist by Borel's theorem. Thus again $d = 2k$.

THEOREM 2.1. *Let Γ be a torsion-free lattice in the connected semisimple linear Lie group L without compact factors. If the dimension d of the symmetric space $M = L/K$ is odd then all $\beta_i(\Gamma)$ are zero. If d is even then all $\beta_i(\Gamma)$ are zero except possibly for the middle dimension $i = \frac{1}{2}d$.*

If the lattice Γ is not asssumed torsion-free then it contains a subgroup Γ_1 of finite index which is torsion-free. Γ_1 is again a lattice in L. The non-vanishing of $\beta_k(\Gamma)$ implies the non-vanishing of $\beta_k(\Gamma_1)$ and thus d must be equal to $2k$:

THEOREM 2.2. *The conclusions of Theorem 2.1 hold for arbitrary lattices in L.*

2.3. As a corollary of Theorem 2.1 one obtains the following known result, which is a special case of the "Singer Conjecture".

THEOREM 2.3. *Let X be a closed Riemannian manifold. If the universal covering $\widetilde{X}$ is a Riemannian symmetric space of non-compact type then all $\beta_i(X)$ are zero except possibly for the middle dimension.*

Indeed $\pi_1(X)$ considered as covering transformation group has a subgroup of finite index which is a cocompact lattice Γ in the 1-component of the isometry group of $\widetilde{X}$. Thus $\beta_i(\Gamma) = \beta_i(X) = 0$ except possibly for the middle dimension.

EXAMPLE 2.4. If X is a closed hyperbolic manifold of dimension d then $\widetilde{X} = \mathbf{H}^d$ is the hyperbolic d-space. The ℓ_2-Betti numbers are zero except for even $d = 2k$ where $\beta_k \neq 0$, see [D].

EXAMPLE 2.5. We consider the non-cocompact lattice $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ in $\mathrm{SL}_n(\mathbf{R})$ for $n \geq 3$, and let Γ_1 be a torsion-free subgroup of finite index in Γ. Although here $X = M/\Gamma_1$ is not compact it was shown by Borel-Serre [B-S] that X is homotopy equivalent to a finite cell complex. The ℓ_2-Betti numbers of Γ_1, whence of Γ, are all zero if d is odd ($n = 4m$ or $n = 4m+3$). They are all zero except possibly for the middle dimension if d is even ($n = 4m+1$ or $n = 4m+2$); but since it is known that for $n \geq 3$ the Euler characteristic of Γ_1 is equal to 0 they are also all zero.

3. Non-vanishing of β_1

3.1. Let again L be a connected semisimple linear Lie group without compact factors, and Γ a lattice in L. If $\beta_1(\Gamma) \neq 0$ then according to Theorem 2.1 the dimension d of the symmetric space $M = L/K$ must be 2. Looking at the list of all L one notices that $d = 2$ for $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{PSL}_2(\mathbf{R})$, and there are no other possibilities.

THEOREM 3.1. *Let Γ be a lattice in the connected semisimple linear Lie group L without compact factors. If the first ℓ_2-Betti number $\beta_1(\Gamma)$ is non-zero then Γ is a lattice in* $\mathrm{SL}_2(\mathbf{R})$ *or* $\mathrm{PSL}_2(\mathbf{R})$. *Thus Γ is a virtual surface group or virtually a non-Abelian free group.*

3.2. We add two immediate corollaries of Theorem 3.1. L and Γ are as in that Theorem.

COROLLARY 3.2. *If Γ is not isomorphic to a lattice in* $\mathrm{PSL}_2(\mathbf{R})$ *then* $\beta_1(\Gamma) = 0$. *This is the case in particular if* $\mathrm{vcd}(\Gamma)$ *is* > 2 *for cocompact* Γ *and* > 1 *for non-cocompact* Γ.

According to Proposition 1.4 the vanishing of $\beta_1(\Gamma)$ implies strong properties of the deficiency of the finitely presented group Γ: namely $\mathrm{def}(\Gamma) \leq 1$ and if $\mathrm{def}(\Gamma) = 1$, then $\mathrm{gdim}(\Gamma) = \mathrm{cd}(\Gamma) \leq 2$.

COROLLARY 3.3. *If Γ is not isomorphic to a lattice in* $\mathrm{PSL}_2(\mathbf{R})$ *then its deficiency* $\mathrm{def}(\Gamma)$ *is* ≤ 1 *and if* $\mathrm{def}(\Gamma) = 1$ *then* $\mathrm{gdim}(\Gamma) = \mathrm{cd}(\Gamma) \leq 2$ *and* Γ *is torsion-free.*

4. Lattices in connected Lie groups

4.1. Let L be a connected Lie group and Γ an (infinite) lattice in L, $\mathrm{Rad}(L)$ the radical of L, i.e. the maximal connected normal solvable subgroup of L.

We recall a general fact (see [A, Proposition 2] or [R, Corollary 8.27]): If $L/\mathrm{Rad}(L)$ has no compact factor then, for any lattice Γ,

$$\Gamma/\Gamma \cap \mathrm{Rad}(L) = \Gamma\,\mathrm{Rad}(L)/\mathrm{Rad}(L)$$

is discrete in $L/\mathrm{Rad}(L)$ whence a lattice.

[The proof that $\Gamma/\Gamma\cap\mathrm{Rad}(L)$ is discrete is based on various results. One first shows that the 1-component C of its closure in $L/\mathrm{Rad}(L)$ is solvable; then that C is normal in $L/\mathrm{Rad}(L)$ (this in turn because a lattice has property (S), see [R, 5.1 ff]); thus $C = 1$, i.e. $\Gamma/\Gamma\cap\mathrm{Rad}(L)$ is discrete.]

We now assume that $\beta_1(\Gamma) \neq 0$. Since the intersection $\Gamma\cap\mathrm{Rad}(L)$ is solvable, whence amenable it must be finite by Proposition 1.3. We write R for the product of $\mathrm{Rad}(L)$ with a suitable compact normal subgroup of L so that $L_1 = L/R$ is semisimple without compact factors. Then $\Gamma\cap R$ is finite and $\Gamma_1 = \Gamma/\Gamma\cap R$ is an infinite lattice in L_1 with $\beta_1(\Gamma_1) \neq 0$.

4.2. The intersection of Γ_1 with the discrete center $Z(L_1)$ must be finite, again by Proposition 1.3. The adjoint representation of L_1 yields a linear Lie group $L_1/Z(L_1)$. It contains the lattice $\Gamma_1/\Gamma_1\cap Z(L_1)$, i.e. the factor group Γ/N of the original lattice Γ by a finite normal subgroup N, and $\beta_1(\Gamma/N) \neq 0$.

4.3. Let Γ_0/N be a torsion-free subgroup of finite index in Γ/N. It is a lattice in a connected semisimple linear Lie group without compact factors; by Theorem 3.1 it is isomorphic to a torsion-free lattice Δ in $\mathrm{PSL}_2(\mathbf{R})$. Since Γ_0 has finite index in Γ and N is finite we get the exact sequence

$$1 \longrightarrow N \longrightarrow \Gamma_0 \longrightarrow \Delta \longrightarrow 1$$

as requested.

THEOREM 4.1. *Let Γ be a lattice in a connected Lie group. If the first ℓ_2-Betti number of Γ is non-zero then Γ is commensurable with a torsion-free lattice Δ in* $\mathrm{PSL}_2(\mathbf{R})$, *i.e. with a surface group of genus ≥ 2 or a non-Abelian free group.*

Thus "in general" the first ℓ_2-Betti number of a lattice in a connected Lie group is zero. As in 3.3 this yields information about the deficiency.

THEOREM 4.2. *If the infinite lattice Γ in the connected Lie group L is not commensurable with a torsion-free lattice in* $\mathrm{PSL}_2(\mathbf{R})$ *then* $\beta_1(\Gamma) = 0$. *The deficiency of Γ is then ≤ 1; and if* $\mathrm{def}(\Gamma) = 1$ *then* $\mathrm{gdim}(\Gamma) = \mathrm{cd}(\Gamma) \leq 2$.

4.4. TWO APPLICATIONS

1) Again we can say that "in general" the deficiency of a lattice in a connected Lie group is ≤ 0. We return to the exceptional cases of deficiency one in Part Two.

2) Let Y be a 4-manifold with fundamental group isomorphic to a lattice Γ in a connected Lie group, Γ not commensurable with a lattice in $\mathrm{PSL}_2(\mathbf{R})$, and let χ be its Euler characteristic and σ its signature. Then (cf [E2]) one has $|\sigma| \leq \chi$.

PART TWO: LATTICES WITH $\beta_1 = 0$ AND DEFICIENCY 1

5. INTRODUCTION

5.1. As shown in Part One an infinite lattice Γ in a connected Lie group L has first ℓ_2-Betti number $\beta_1(\Gamma) = 0$, whence deficiency ≤ 1, except if Γ is commensurable with a torsion-free lattice in $\mathrm{PSL}_2(\mathbf{R})$, i.e. with a surface group of genus ≥ 2 or a free non-Abelian group. The objective of Part Two is to show that lattices with $\beta_1 = 0$ and def $= 1$ are exceptional, and to give a list of these. For any other infinite lattice in a connected Lie group, not commensurable with a surface group of genus ≥ 2 or a free non-abelian group, the deficiency is ≤ 0.

5.2. The crucial fact for the lattices Γ considered here is that they are of geometric and cohomological dimension ≤ 2 whence torsion-free and have a finite $K(\Gamma,1)$-complex; and that their Euler characteristic is 0. This strongly limits the possibilities for the respective Lie groups.

6. THE COHOMOLOGY DIMENSION OF Γ AND THE SOLVABLE CASE

6.1. As shown in Section 1 (in particular in Proposition 1.4), the conditions $\beta_1(\Gamma) = 0$ and $\mathrm{def}(\Gamma) = 1$ imply, actually for any finitely presented infinite group, that $\beta_2(\Gamma) = 0$, that the 2-dimensional presentation complex, for a presentation with deficiency 1, is a $K(\Gamma,1)$, and that the Euler characteristic $\chi(\Gamma) = 0$. Thus $\mathrm{gdim}(\Gamma) = \mathrm{cd}(\Gamma) \leq 2$ and Γ is torsion-free. Note that conversely $\mathrm{gdim}(\Gamma) \leq 2$ and $\beta_1(\Gamma) = 0$ and $\chi(\Gamma) = 0$ imply $\mathrm{def}(\Gamma) = 1$.

6.2. We first consider the cases $\mathrm{cd}(\Gamma) = 0$ and 1.

Since Γ is infinite $\mathrm{cd}(\Gamma)$ is $\neq 0$. If $\mathrm{cd}(\Gamma) = 1$ then Γ is a free group by Stallings' Theorem 1.5. As $\beta_1 = 0$, Γ must be equal to $\mathbf{Z}$ and $L = \mathbf{R}$.

In what follows, Γ is always of cohomology dimension 2.

6.3. If L is solvable then Γ is cocompact. Thus in our case L/Γ is a closed $K(\Gamma, 1)$-manifold of dimension 2 with solvable fundamental group. The only possibilities for that manifold are the 2-torus and the Klein bottle; i.e. $\Gamma = \mathbf{Z}^2$ or $\Gamma =$ fundamental group of the Klein bottle, and $L = \mathbf{R}^2$.

7. The semisimple case

7.1. Let L be a connected semisimple Lie group without compact factors, $M = L/K$ the symmetric space ($K =$ maximal compact subgroup), Γ a torsion-free lattice in L. We recall that M is the universal covering of a $K(\Gamma, 1)$-manifold.

In order to get examples for lattices Γ with $\mathrm{gdim}(\Gamma) = \mathrm{cd}(\Gamma) = 2$ and with $\beta_1(\Gamma) = 0$ we look for cases where the dimension $d(M)$ is 3, the lattice being torsion-free and non-cocompact. From 3.1 it follows that $\beta_1(\Gamma) \neq 0$ would imply $d(M) = 2$; thus $\beta_1(\Gamma)$ is indeed zero. Since $d(M)$ is odd the Euler characteristic $\chi(\Gamma)$ is zero and this guarantees (see 6.1) that $\mathrm{def}(\Gamma) = 1$. To prove that $\chi(\Gamma) = 0$ one can either use Theorem 3.1 which tells that all ℓ_2-Betti numbers of Γ are zero. A different argument (not using Borel-Wallach etc) is to note that the Euler-Poincaré measure μ in the case of odd d is identically 0 [S, paragraph 3] and $\chi(\Gamma)$, the integral of μ over L/Γ is zero.

If $\dim(M) = 3$ and Γ not cocompact, then $\mathrm{cd}(\Gamma) \leq 2$. But $\mathrm{cd}(\Gamma) = 1$ would mean $\Gamma = \mathbf{Z}$ which here is not possible. Thus $\mathrm{cd}(\Gamma) = 2$.

7.2. Such an example is $L = \mathrm{PSL}_2(\mathbf{C})$, considered as a real Lie group; its rank ℓ is 1, its dimension is 6 and the dimension of K is 3. The torsion-free non-cocompact lattices in L fulfill all the required properties above.

7.3. Another example of a similar kind is $\widetilde{P}$, the universal covering of $P = \mathrm{PSL}_2(\mathbf{R})$. It contains non-cocompact lattices; if torsion-free they act freely by translation on $\widetilde{P}$, of dimension 3, and they have the required properties.

7.4. We will show that these two examples are (essentially) the only semisimple connected Lie groups admitting lattices of the required type.

Observe first that $\widetilde{P}$ and $\mathrm{PSL}_2(\mathbf{C})$ are the only such examples of rank $\ell = 1$ with $d(M) = 3$. Thus we consider cases of rank $\ell = 2$ or higher. We show below that the only possibility to be examined is the special case $L = \mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ or its finite coverings (same arguments for $\mathrm{SL}_2(\mathbf{R})$ instead of $\mathrm{PSL}_2(\mathbf{R})$); here $d = 4$. If Γ is a non-cocompact lattice L with $\beta_1(\Gamma) = 0$ and $\mathrm{def}(\Gamma) = 1$ then (see 6.1) $\chi(\Gamma)$ would have to be zero.

However, in this case it is > 0 since the Euler-Poincaré measure μ is positive (the ranks of L and of K are equal [S, p. 136]). Thus this special case is ruled out. We still have to prove that there are no other possibilities for L of rank $\ell = 2$ or higher.

If Γ is an irreducible lattice in L then it is arithmetic. According to Borel-Serre [B-S] $\mathrm{cd}(\Gamma)$ is then $\geq d - \ell$ which is, except for the "special case" above, > 2. If Γ is reducible, i.e. the direct product of irreducible lattices in semisimple or simple factors of L then their cohomology dimension is again > 2. Thus these cases (which include the infinite coverings of the special case above) are eliminated.

THEOREM 7.1. *An infinite lattice Γ in a connected semisimple Lie group with $\beta_1(\Gamma) = 0$ and $\mathrm{def}(\Gamma) = 1$ is isomorphic to a non-cocompact lattice either in $\mathrm{PSL}_2(\mathbf{C})$ or in the universal covering of $\mathrm{PSL}_2(\mathbf{R})$.*

7.5. For later use we look more closely at non-cocompact lattices Γ in $\widetilde{P}$. We claim that the image of Γ under the covering map $p\colon \widetilde{P} \to P$, with central kernel $\mathbf{Z}$, remains discrete. To prove this consider Γ as a lattice in $L = \mathbf{R} \times_{\mathbf{Z}} \widetilde{P}$. Then L modulo its radical $\mathbf{R}$ is $\widetilde{P}/\mathbf{Z} = P$, and according to 4.1 the image of Γ is indeed discrete. The intersection of Γ with the kernel of p cannot be 1 since this would imply that p maps Γ isomorphically onto a lattice in P and $\beta_1(\Gamma)$ would be $\neq 0$. Thus the intersection is isomorphic to $\mathbf{Z}$ and $\Gamma/\mathbf{Z}$ is isomorphic to a non-cocompact lattice in $\mathrm{PSL}_2(\mathbf{R})$. Therefore the non-cocompact lattices in $\widetilde{P}$ are central extensions of non-cocompact lattices in $\mathrm{PSL}_2(\mathbf{R})$ by $\mathbf{Z}$.

7.6. We remark that a group Γ which is a central extension as above has $\beta_1(\Gamma) = 0$ (and, if torsion free, $\mathrm{gdim}(\Gamma) = 2$ and $\chi(\Gamma) = 0$ whence $\mathrm{def}(\Gamma) = 1$). These are exactly the groups isomorphic to non-cocompact lattices in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$:

PROPOSITION 7.2. *A group Γ is a central extension of a non-cocompact lattice Γ_0 in $\mathrm{PSL}_2(\mathbf{R})$ by $\mathbf{Z}$ if and only if it is a non-cocompact lattice in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$.*

Proof. Let Γ be such an extension. Note that Γ_0 being virtually free its ordinary cohomology group $H^2(\Gamma_0; \mathbf{R})$ is 0. Thus under the imbedding of $\mathbf{Z}$ into $\mathbf{R}$ the extension splits into a direct product $\mathbf{R} \times \Gamma_0$ and Γ is isomorphic to a non-cocompact lattice in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$.

On the other hand the radical of $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$ is $\mathbf{R}$; the projection of a lattice Γ to the second factor remains discrete and thus is a lattice Γ_0 in $\mathrm{PSL}_2(\mathbf{R})$, and the intersection of Γ with $\mathbf{R}$ must be isomorphic to $\mathbf{Z}$. It follows that Γ is a central extension of Γ_0 by a lattice $\mathbf{Z}$ in the first factor.

COROLLARY 7.3. *Non-cocompact lattices in $\widetilde{P}$ can be imbedded as non-cocompact lattices in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$.*

8. THE GENERAL CASE

8.1. As in Section 4 let L be an arbitrary connected Lie group, R its radical times a suitable compact normal subgroup so that L/R is semisimple without compact factors.

If Γ is a lattice in L let Γ_1 be its intersection with R. Then $\Gamma/\Gamma_1 = \Delta$ is a lattice in L/R.

We now assume that $\beta_1(\Gamma) = 0$ and $\mathrm{def}(\Gamma) = 1$; then $\mathrm{cd}(\Gamma) = 2$. The lattice Δ need not be torsion-free but has a torsion-free subgroup of finite index. The virtual cohomology dimension $\mathrm{vcd}(\Delta)$ is finite (it is the cohomology dimension of any torsion-free subgroup of finite index). Therefore the additivity of the cohomology dimensions of the torsion free subgroups of Δ, of Γ and of its normal subgroup Γ_1 yields

$$\mathrm{vcd}(\Gamma) = \mathrm{vcd}(\Gamma_1) + \mathrm{vcd}(\Delta).$$

Now $\mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma) = 2$ and $\mathrm{vcd}(\Gamma_1) = \mathrm{cd}(\Gamma_1) \leq 2$ since Γ and Γ_1 are torsion-free.

[Concerning the additivity of cohomology dimensions for group extensions $G/N = Q$ we remark that one has to assume that $\mathrm{cd}(Q)$ is finite – which is the case here for the torsion-free subgroup of Δ – and that in general $\mathrm{cd}(G) \leq \mathrm{cd}(N) + \mathrm{cd}(Q)$. However here the kernel Γ_1 of the extension, being the fundamental group of a closed manifold, fulfills the sufficient condition for equality.]

8.2. If $\mathrm{cd}(\Gamma_1) = 2$ then $\mathrm{vcd}(\Delta) = 0$. This means that Δ is finite; since L/R has no compact factor $\Delta = 1$ and $L/R = 1$. Thus $\Gamma = \Gamma_1$ and we are in the solvable case (Section 6). If $\mathrm{cd}(\Gamma_1) = 0$ then $\Gamma_1 = 1$ and we are in the semisimple case (Section 7).

8.3. The only new case is $\mathrm{cd}(\Gamma_1) = 1$; then Γ_1 is free whence equal to $\mathbf{Z}$; the radical of L must be $\mathbf{R}$. As for Δ we get $\mathrm{vcd}(\Delta) = 1$, i.e. Δ

is virtually free, actually virtually non-abelian free since it is a lattice in a semisimple Lie group. As a non-cocompact lattice it lies in $P = \mathrm{PSL}_2(\mathbf{R})$ or perhaps in a finite covering $\bar{P}$ of P. If it lies in P then the Lie group must be, apart possibly from a compact normal subgroup, $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$, containing Γ as a non-cocompact lattice. If it lies in $\bar{P}$ the kernel of its projection to P is finite cyclic; the kernel of the projection of Γ to P via $\bar{P}$ is a finite extension of $\mathbf{Z}$ and torsion free, whence equal to $\mathbf{Z}$. Thus Γ is again an extension of a non-cocompact lattice Γ_0 in P by $\mathbf{Z}$, lying in the same product and necessarily central.

Thus the only new example provided by the "general case" are the non-cocompact lattices in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$. We recall that the non-cocompact lattices in $\widetilde{P}$ (Corollary 7.3) are also central extensions of the same kind and can be imbedded in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$. So there is no need to mention $\widetilde{P}$ in our final list.

8.4. We summarize the results.

THEOREM 8.3. *Lattices in a connected Lie group with vanishing β_1 and deficiency one are isomorphic either to the (cocompact) solvable cases, the fundamental groups of the circle or the 2-torus or the Klein bottle; or to the non-cocompact lattices in* $\mathrm{PSL}_2(\mathbf{C})$ *or in* $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$. *For all other lattices in a connected Lie group, not commensurable with a surface group of genus ≥ 2 or a non-abelian free group, the deficiency is ≤ 0.*

9. KNOT GROUPS AND LATTICES

9.1. A knot group G (see [Ro]), the fundamental group of the (closed) complement C of a classical non-trivial knot, has deficiency one and vanishing first ℓ_2-Betti number [L-L]. The complement C is aspherical, whence a $K(G,1)$-manifold-with-boundary. The question as to whether a knot group is a lattice in a connected Lie group is the motivation for Part Two although the results go beyond that problem.

The solvable cases of Theorem 8.3 do not occur as knot groups, apart from the group $\mathbf{Z}$ of the trivial knot. So knot groups can be lattices only in $\mathrm{PSL}_2(\mathbf{C})$ or in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$.

The torus knot groups are isomorphic to lattices in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$ since they are central extensions by $\mathbf{Z}$ of virtual finitely generated non-abelian free groups (cf Proposition 7.2). This agrees with Thurston geometrization which tells that the interior of the knot complement, in this case, is a Seifert fibering.

Since the torus knot groups are the only knot groups with non-trivial center the groups of all other knots cannot be lattices in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$.

9.2. So we now consider knots which are not torus knots; their groups can be lattices in $\mathrm{PSL}_2(\mathbf{C})$ only. As for $\mathrm{PSL}_2(\mathbf{C})$, it is the isometry group of hyperbolic 3-space $\mathbf{H}^3$. The knot complement C is a Haken 3-manifold with zero-Euler characteristic. It is atoroidal precisely if the respective knot does not have a companion; we use here "companion" in the sense of non-trivial companion. Indeed the boundary torus of a regular neighborhood of a companion would be a non-boundary-parallel incompressible surface in C. Thus by Thurston's Hyperbolization Theorem the interior of the complement of a knot without companion can be given a hyperbolic structure with finite volume coming from $\mathbf{H}^3/G$. In other words that knot group G is a lattice in $\mathrm{PSL}_2(\mathbf{C})$ and the interior of C can be identified with the open manifold $\mathbf{H}^3/G$.

Concerning the notion of companion see [Ro, p. 111]. For concepts related to the Hyperbolization Theorem and to properties of 3-manifolds we refer to [K].

9.3. If the knot has a companion then the knot complement is not atoroidal and its interior does not admit a hyperbolic structure [K, Cor.4.63]. It follows that G cannot be a lattice in (the only remaining possibility) $\mathrm{PSL}_2(\mathbf{C})$. We sketch the proof: If G is such a lattice then $\mathbf{H}^3/G$ is a $K(G,1)$-manifold as well, thus homotopy equivalent to C. The homotopy equivalence can be turned into a diffeomorphism mapping $\mathbf{H}^3/G$ to the interior of the knot complement C which thus would receive a hyperbolic structure with finite volume.

THEOREM 9.1. *Torus knot groups are lattices in* $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$. *As for other groups of knots, those of knots without companion are lattices in* $\mathrm{PSL}_2(\mathbf{C})$, *and those of knots with companion are not lattices in any connected Lie group.*

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