The Singer conjecture

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THE SINGER CONJECTURE

by Beno ECKMANN

This is a conjecture and a series of discussions concerning aspherical spaces and ℓ_2-Betti numbers (or ℓ_2-homology groups). The conjecture has appeared in various slightly different versions. We will consider a quite general formulation. But first we make a very short description of the two concepts "aspherical" and "ℓ_2". The spaces considered here are always (of the homotopy type of) cell-complexes.

1. ASPHERICAL SPACES. A connected space X is called *aspherical* if all (continuous) images of spheres S^n, $n \geq 2$ in X are contractible. This is equivalent to the vanishing of all homotopy groups $\pi_n(X)$, $n \geq 2$. It implies the same property for the universal covering $\widetilde{X}$ of X, and since $\pi_1(\widetilde{X}) = 0$ the space $\widetilde{X}$ is contractible to a point. Conversely the contractibility of $\widetilde{X}$ implies that X is aspherical.

If two aspherical spaces have isomorphic fundamental groups then they are homotopy equivalent. Thus the (homotopical) topological invariants of the two spaces are the same and depend on the fundamental group only. This concerns in particular all kinds of homology. Homological algebra has been developed to compute such invariants directly from the given group. However, in many cases it is preferable to use the aspherical spaces. Note that for any group G there exist aspherical spaces having G as fundamental group (called *classifying spaces* for G).

2. ℓ_2-HOMOLOGY. ℓ_2-homology and ℓ_2-Betti numbers differ from ordinary homology and Betti numbers with real coefficients by using infinite chains which constitute Hilbert spaces. We explain this in a few words. The procedure is essentially the same, with slight but important differences.

We consider an n-dimensional finite cell-complex X and its universal covering $\widetilde{X}$ which in general will be infinite. The square-summable chains of $\widetilde{X}$, real linear combinations of the cells of $\widetilde{X}$, form Hilbert spaces C_i, $i = 0, \dots, n$ on which the fundamental group $G = \pi_1(X)$ operates by permutation of all cells above a cell of X, compatibly with the ordinary boundary operator $\partial\colon C_i \longrightarrow C_{i-1}$. Actually above a cell of X a chain $\in C_i$ is an element of $\ell_2 G$, the square summable real linear combinations of the elements of G and $C_i = \ell_2 G^{\alpha_i}$, where α_i is the number of i-cells of X.

Homology of the chain complex $C = \{C_i\}$ would simply be ordinary homology of X with $\ell_2 G$ coefficients. Here, however, comes the main difference between ordinary and ℓ_2-homology. H_i is defined as the kernel of ∂ in C_i modulo the closure of ∂C_{i+1}. This yields a Hilbert space structure on H_i. Moreover H_i can easily be identified with the subspace of C_i consisting of "harmonic" chains that are both cycles and cocycles (cohomology is defined by the dual δ of ∂ exactly as homology, and they are isomorphic).

The group G operates on H_i by isometries and the embedding of H_i in C_i is isometric and G-equivariant. It has a von Neumann dimension which is a non-negative real number, namely the trace of the projection operator $C_i \longrightarrow C_i$ with image H_i. That dimension is the *ℓ_2-Betti number* β_i of X. In many but not in all properties the ℓ_2-Betti numbers behave like ordinary Betti numbers. In particular they are homotopy invariant and they compute the elementary Euler characteristic

$$\chi(X) = \beta_0 - \beta_1 + - \cdots + (-1)^n \beta_n$$

exactly as the ordinary Betti numbers do.

3. VANISHING OF ℓ_2-BETTI NUMBERS. It often happens that ℓ_2-Betti numbers are $= 0$ in cases where ordinary Betti numbers are not $= 0$. This may have interesting consequences. An easy case is $\beta_0 = 0$ for any infinite group G. We consider here only groups with finite cell-complexes as classifying space. If the group G is infinite amenable then all ℓ_2-Betti numbers are $= 0$; thus the Euler characteristic is $= 0$. This is true by a much deeper result if the group contains an infinite amenable normal subgroup (without any finiteness assumption).

An example of the group-theoretic implication: if $\beta_1 = 0$ then the deficiency of the group presentation is ≤ 1.

The Singer Conjecture is not concerned with group-theoretic properties but with cases where for an aspherical closed manifold all ℓ_2-Betti numbers are $=0$ with one possible exception. In Section 4, we describe the basic example coming from geometry.

4. SYMMETRIC MANIFOLD. We now assume that X is a compact Riemannian manifold and that its universal covering $\widetilde{X}$ is the symmetric space of a semi-simple linear Lie group L without compact factors, $\widetilde{X} = L/K$ where K is a maximal compact subgroup. The fundamental group $\pi_1(X)$ operates on $\widetilde{X}$ and by the ℓ_2-version of the de Rham–Hodge theorem, the cohomology of $\widetilde{X}$ is isomorphic (relative to $\pi_1(X)$) to the group of harmonic square-integrable differential forms on $\widetilde{X}$. If $\beta_k(X)$ is $\neq 0$ then $\widetilde{X}$ contains non-zero harmonic k-forms. By the Borel–Wallach theorem this is possible only in the middle dimension of X : all $\beta_k(X) = 0$ except possibly for the middle dimension.

Can one get rid of the symmetry assumption and of the differential geometry involved? The unproved general idea is that this is the case:

CONJECTURE 29.1. *Let X be a closed aspherical Riemannian manifold. All $\beta_k(X)$ are $=0$ except possibly for the middle dimension.*

The above conjecture has an interesting relation to a very old conjecture of Heinz Hopf concerning the Euler characteristic $\chi(X)$ of a closed aspherical Riemannian manifold of even dimension $2d$. That conjecture, of course, had nothing to do with ℓ_2-homology. However if the above conjecture holds then the Hopf conjecture predicting $(-1)^d\chi(X) \geq 0$ follows immediately. Moreover the 'strict' version of the Hopf conjecture prompting $(-1)^d\chi(X) > 0$ when X has everywhere (strictly) negative sectional curvatures would be equivalent to $\beta_d(X) \neq 0$. This has been proved in special cases but is still not known in general.

B. Eckmann
FIM-ETHZ
Rämistrasse 101
CH-8092 Zürich
Switzerland
e-mail: eckman@math.ethz.ch