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NINE PROBLEMS ABOUT (CO-)LOCALIZATIONS

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INTRODUCTION

In what follows we denote by X, Y etc. either groups or pointed spaces, assuming them to be either CW-complexes or simplicial sets satisfying the usual Kan conditions when needed. It is well known that a map $\phi\colon X \to Y$ has the form $X \to L_f Y$ of the canonical co-augmentation map for *some* localization functor L_f if and only if it induces an equivalence on the appropriate mapping objects: (i.e. sets of maps with the appropriate extra structure on them. For spaces these are pointed function complexes, for groups these are sets of group maps.)

$$map_*(Y,Y) \to map_*(X,Y)\,.$$

Such a map ϕ will be called here a *localization map*. The notation map_* denotes either the set of all group-maps or the space of all *pointed maps*.

Similarly, a map $\psi\colon V \to W$ is called *cellular* if it is of the form of the canonical augmentation map for the cellularization functor in the relevant category: $cell_A W \to W$. It is not hard to see that a map is cellular if and only if it induces an equivalence $map_*(V,V) \to map_*(V,W)$. In that case $V = cell_A W$ is equivalent to $cell_V W$.

Given the above concepts, most of the following problems-conjectures are elementary in their formulations. But some have proven surprisingly difficult to confirm or negate. I will not dwell here on their implications, the problems seem sufficiently simple minded and attractive as they stand. Most of them can be generalized in various ways, to yield statements in other categories. Some progress and results of similar nature on these and related problems is indicated in the references cited below. On the basis of special cases, one may expect positive answers for these questions, maybe under mild additional assumptions, save maybe problems numbers 3 and 4.

NINE OPEN PROBLEMS

PROBLEM 1. Prove that any localization $P \to L_f P$ of a finite p-group P is a surjective map, in particular the localization is a finite p-group. This is known for groups of nilpotent class 3, by a result of M. Aschbacher.

PROBLEM 2. More generally, any localization $L_f N$ of a nilpotent group is a nilpotent group.

PROBLEM 3. Is it true that, under mild restrictions on f and A, the composite functors $cell_A \circ L_f$ and $L_f \circ cell_A$ are idempotent functors ?

PROBLEM 4. In certain cases the localization of a principal fibration sequence $G \to E \to B$ should be a principal fibration sequence, say if one localizes with respect to a suspension map. In general the fibration is not preserved; however, its principal nature is supposed to be preserved under some restrictions. If true this would be in line with the well-known Bousfield–Kan fibre lemma about R_∞ (for G connected) where the fibration is actually preserved and with the observation that it holds for the Postnikov-section functor $L_f = P_n$. For the cellular approximation functor $cell_A \equiv CW_A$ this is always true [2].

PROBLEM 5. The localization of any 1-connected space is 1-connected. A weaker version: a universal covering projection $\widetilde{X} \to X$ is a localization map only if it is an equivalence, namely, if X is 1-connected.

PROBLEM 6. Any localization of a polyGEM is a polyGEM. The localization of an n-connected Postnikov stage ($n \geq 1$) is an n-connected Postnikov stage.

PROBLEM 7. A special case of Problem 3: Let P_1 be the first Postnikov section functor. Let $X = cell_A K(\pi, 1)$ where π is a discrete group and A a pointed space. Is it true that one has an equivalence:

$$X \cong cell_A P_1 X .$$

It is not hard to verify that the equivalence hold in case A is a two-dimensional complex or in case A is an acyclic space.

PROBLEM 8. Any localization and cellularization of a space X (we may need to assume it is an n-Postnikov section) all whose homotopy groups are p-torsion, is also such a space.

PROBLEM 9. The cellularization of a finite Postnikov stage with finite homotopy groups, is a space whose homotopy groups are finite groups. This is true in the category of groups, for finite groups. But for spaces, it is not clear even for a $K(\pi, 1)$ with π a finite group.

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