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THE FRIEDLANDER–MILNOR CONJECTURE

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The conjecture of the title of this note has resisted 40 years of effort and remains not only unsolved but also lacking in a plausible means of either proof or counter-example.

The original form of this conjecture is one I struggled with during my days at Princeton in the early 1970's:

CONJECTURE 32.1. *Let $G(\mathbf{C})$ be a complex reductive algebraic group and let $G(\mathbf{C})^\delta$ denote this group viewed as a discrete group. Then the map on classifying spaces of the continuous (identity) group homomorphism*

$$i \colon G(\mathbf{C})^\delta \to G(\mathbf{C})$$

induces an isomorphism in cohomology with finite coefficients $\mathbf{Z}/n$ for any $n \geq 0$:

$$i^* : H^*(BG(\mathbf{C}), \mathbf{Z}/n) \overset{i^*}{\simeq} H^*(G(\mathbf{C})^\delta, \mathbf{Z}/n).$$

Conjecture 32.1 is easily seen to be true for a torus (i.e., $G = \mathbf{G}_m^{\times r}$ for some $r > 0$), but even the simplest non-trivial case (that of $G = \mathrm{SL}_2$) remains inaccessible.

Guido and I published 5 papers together, all in some sense connected with this conjecture. We used the integral form $G_{\mathbf{Z}}$ of a complex reductive algebraic group (which is a group scheme over $\operatorname{Spec} \mathbf{Z}$) in order to form the group $G(F)$ of points of G with values in a field F. Most of our joint work investigated various relations between $G(\mathbf{C})$ and $G(F)$, the case $F = \overline{\mathbf{F}}_p$ (the algebraic closure of a prime field $\mathbf{F}_p$) being of special interest.

One knows from considerations of étale cohomology that the cohomology of $BG(\mathbf{C})$ with $\mathbf{Z}/n$ coefficients is naturally isomorphic to that of the étale homotopy classifying space of the algebraic group G_F for F algebraically closed of characteristic $p \geq 0$:

$$H^*(BG(\mathbf{C}), \mathbf{Z}/n) \simeq H^*((BG_F)_{\mathrm{et}}, \mathbf{Z}/n), \quad \text{provided that } (p,n) = 1.$$

This enables one to construct a map $H^*(BG(\mathbf{C}), \mathbf{Z}/n) \longrightarrow H^*(G(F), \mathbf{Z}/n)$ relating the cohomology with mod-n coefficients of the classifying space of $G(\mathbf{C})$ with the cohomology with mod-n coefficients of the discrete group $G(F)$ for any field F.

The following is a generalization of Conjecture 32.1, one that appears likely to be true if and only if Conjecture 32.1 is valid.

CONJECTURE 32.2. *Let $G(\mathbf{C})$ be a complex reductive algebraic group, let $n > 0$ be a positive integer, and let p denote either 0 or a prime which does not divide n. Then for any algebraically closed field F of characteristic p, the comparison of the cohomology of $BG(\mathbf{C})$ and $G(F)$ determines an isomorphism*

$$H^*(G(F), \mathbf{Z}/n) \simeq H^*(BG(\mathbf{C}), \mathbf{Z}/n).$$

In our first paper together [1], Guido and I began our investigation of "locally finite approximations" of Lie groups. We also formulated the following conjecture and proved it equivalent to Conjecture 32.2.

CONJECTURE 32.3. *Let F be an algebraically closed field of characteristic $p \geq 0$ and let $n > 0$ be a positive integer not divisible by p if $p > 0$. Then Conjecture 32.2 is valid for $G(F)$ if and only for every $0 \neq x \in H^*(G(F), \mathbf{Z}/n)$, there exists some finite subgroup $\pi \subset G(F)$ such that x restricts non-trivially to $H^*(\pi, \mathbf{Z}/n)$.*

The most familiar form of the "Friedlander–Milnor Conjecture" is that formulated by John Milnor in [2]. In that paper, Milnor verifies this conjecture for solvable groups.

CONJECTURE 32.4. *Let G be a Lie group with finitely many components and let G^δ denote the same group now viewed as a discrete group. Then for any integer $n > 0$, the continuous (identity) map $i \colon G^\delta \to G$ induces an isomorphism on cohomology with mod-n coefficients:*

$$i^* \colon H^*(BG, \mathbf{Z}/n) \overset{i^*}{\simeq} H^*(G^\delta, \mathbf{Z}/n).$$

We remark that the most substantial progress to date on these conjectures is due to Andrei Suslin, who proves a "stable" version of Conjectures 32.1 and 32.2 in [3].

REFERENCES

[1] Friedlander, E. and G. Mislin. Cohomology of classifying spaces of complex Lie groups and related discrete groups. *Comment. Math. Helv. 59* (1984), 347–361.

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