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Autor(en): **Geoghegan, Ross**

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THE FUNDAMENTAL GROUP AT INFINITY

by Ross GEOGHEGAN

Let G be a finitely presented group which has one end. There are three flavors of the question: homological, homotopical, and geometric.

THE HOMOLOGICAL FLAVOR

QUESTION 33.1. *Is it true that the abelian group $H^2(G, \mathbf{Z}G)$ is free?*

- REMARKS 33.2. (i) $H^0(G, \mathbf{Z}G)$ and $H^1(G, \mathbf{Z}G)$ are trivial.
(ii) $H^2(G, \mathbf{Z}G)$ is either trivial, or is infinite cyclic, or is an infinitely generated abelian group ([5]).
(iii) $H^n(G, \mathbf{Z}G)$ need not be free abelian when $n > 2$ [1], [4].
(iv) $H^2(G, \mathbf{Z}G)$ need not be free abelian when G is only finitely generated.

Perhaps FP_2 could replace “finitely presented” in Question 33.1.

THE HOMOTOPICAL FLAVOR

Let X be any (one-ended) complex on which G acts cocompactly as a group of covering transformations.

QUESTION 33.3. *Is it true that the “fundamental group at infinity” of X is semistable (aka Mittag-Leffler)?*

An inverse sequence of groups $\{G_r\}$ is *semistable* or *Mittag-Leffler* if, given any n , the sequence of images of the groups G_{n+k} in G_n is eventually constant. We choose a proper ray $\omega: [0, \infty) \rightarrow X$ and a filtration of X by finite subcomplexes K_n . By reparametrizing ω we can assume $\omega([r, \infty)) \subset X - K_r$ for all r . Let G_n denote the fundamental group of the complement of K_n based at $\omega(n)$, and let $f_n: G_{n+1} \rightarrow G_n$ be induced by inclusion using change of base point along ω . Question 33.3 asks if this $\{G_r\}$ is semistable.

REMARKS 33.4. (i) The answer only depends on G , not on X nor on the filtration nor on the base ray; so I can rephrase the homotopical question as

QUESTION 33.5. *Is G semistable at infinity?*

(ii) The answer is known to be *yes* for many classes of groups. For example, all of the following imply that G is semistable at infinity:

- G sits in the middle of a short exact sequence of infinite groups where the kernel is finitely generated [7].
- G is a one-relator group [9].
- G is the fundamental group of a graph of groups whose vertex groups are finitely presented and semistable at infinity, and whose edge groups are finitely generated [8].

(iii) There are positive answers coming from topology. Assume X admits a Z-set compactifying boundary. Then the answer is *yes* if and only if this (connected) boundary has semistable pro- π_1 in the sense of shape theory (the technical term is “pointed 1-movable”); examples are Coxeter groups [3]. This π_1 -condition holds if the boundary is locally connected; examples are hyperbolic groups [2], [10].

(iv) The answer is unknown for CAT(0) groups (as far as I know).

The homological Question 33.1 is equivalent to:

QUESTION 33.6. *Is it true that the inverse sequence of integral first homology groups of the spaces $X - K_n$ is semistable?*

Thus Question 33.1 is the abelianized version of Question 33.3, and is perhaps more likely to have a positive answer.

THE GEOMETRIC FLAVOR

QUESTION 33.7. *Is it true that any two proper rays in X are properly homotopic?*

This is so deliciously simple and “right” that it hardly needs comment³⁾ except to say that it is *equivalent* to Question 33.3 [7].

³⁾ My book [6] contains a much more detailed account of what I summarize here.

FINAL REMARK. There are lots of contractible locally finite 2-dimensional complexes X having one end whose fundamental groups at infinity are not semistable; for example the infinite inverse mapping telescope S associated with a dyadic solenoid (suitably coned off to make it contractible). The problem is to know if any of these admit a cocompact, free and properly discontinuous group action. We know that S does not admit such an action.

REFERENCES

- [1] BESTVINA, M. and G. MESS. The boundary of negatively curved groups. *J. Amer. Math. Soc.* 4 (1991), 469–481.
- [2] BOWDITCH, B.H. Splittings of finitely generated groups over two-ended subgroups. *Trans. Amer. Math. Soc.* 354 (2002), 1049–1078.
- [3] DAVIS, M.W. Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Ann. of Math.* (2) 117 (1983), 293–324.
- [4] —— The cohomology of a Coxeter group with group ring coefficients. *Duke Math. J.* 91 (1998), 297–314.
- [5] FARRELL, F.T. The second cohomology group of G with \mathbb{Z}_2G coefficients. *Topology* 13 (1974), 313–326.
- [6] GEOGHEGAN, R. *Topological Methods in Group Theory*. Graduate Texts in Mathematics 243. Springer-Verlag, 2008.
- [7] MIHALIK, M.L. Semistability at the end of a group extension. *Trans. Amer. Math. Soc.* 277 (1983), 307–321.
- [8] MIHALIK, M.L. and S.T. TSCHANTZ. *Semistability of Amalgamated Products and HNN-Extensions*. Mem. Amer. Math. Soc. 98. Amer. Math. Soc., 1992.
- [9] MIHALIK, M.L. and S.T. TSCHANTZ. One relator groups are semistable at infinity. *Topology* 31 (1992), 801–804.
- [10] SWARUP, G.A. On the cut point conjecture. *Electron. Res. Announc. Amer. Math. Soc.* 2 (1996), 98–100.

Ross Geoghegan

Department of Mathematical Sciences
 Binghamton University (SUNY)
 Binghamton, NY 13902-6000
 USA
e-mail: ross@math.binghamton.edu