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ℓ^p-HOMOLOGY OF ONE-RELATOR GROUPS

by Peter A. LINNELL

Let G be a group and let

$$\cdots \longrightarrow \mathbf{C}G^{e_{n+1}} \xrightarrow{d_n} \mathbf{C}G^{e_n} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathbf{C}G^{e_1} \xrightarrow{d_0} \mathbf{C}G \longrightarrow \mathbf{C} \longrightarrow 0$$

be a free $\mathbf{C}G$-resolution of $\mathbf{C}$. Let $1 \leq p < \infty$ and for $n \geq 0$, let

$$\begin{aligned} d_*^n &: \mathbf{C}G^{e_{n+1}} \otimes_{\mathbf{C}G} \ell^p(G) \longrightarrow \mathbf{C}G^{e_n} \otimes_{\mathbf{C}G} \ell^p(G)\,, \\ d_n^* &: \operatorname{Hom}_{\mathbf{C}G}(\mathbf{C}G^{e_n}, \ell^p(G)) \longrightarrow \operatorname{Hom}_{\mathbf{C}G}(\mathbf{C}G^{e_{n+1}}, \ell^p(G)) \end{aligned}$$

be the maps induced by d_n; for convenience, we let $d_*^{-1} = d_{-1}^* = 0$. Then one has the usual homology and cohomology groups

$$\begin{aligned} H_n(G, \ell^p(G)) &= \ker d_*^{n-1} / \operatorname{im} d_*^n\,, \\ H^n(G, \ell^p(G)) &= \ker d_n^* / \operatorname{im} d_{n-1}^*\,, \end{aligned}$$

which we shall call the (unreduced) *ℓ^p-homology* and *cohomology groups* of G respectively. In the case when all the e_n are finite,

$$\mathbf{C}G^{e_n} \otimes_{\mathbf{C}G} \ell^p(G) \cong \ell^p(G)^{e_n} \cong \operatorname{Hom}_{\mathbf{C}G}(\mathbf{C}G^{e_n}, \ell^p(G))\,,$$

so one can also define the *reduced ℓ^p-homology* and *cohomology groups* of G:

$$\begin{aligned} \overline{H}_n(G, \ell^p(G)) &= \ker d_*^{n-1} / \overline{\operatorname{im} d_*^n}\,, \\ \overline{H}^n(G, \ell^p(G)) &= \ker d_n^* / \overline{\operatorname{im} d_{n-1}^*}\,, \end{aligned}$$

where $\overline{}$ indicates the closure in $\ell^p(G)^{e_n}$. The first ℓ^p-cohomology groups (reduced and unreduced) have been studied extensively recently by Bekka, Bourdon, Martin, Valette and others. Also Kappos has interesting results on general reduced homology and cohomology groups.

Let us concentrate now on the case where G is a finitely generated torsion-free one-relator group. Let d denote the number of generators of G. Then we have a $\mathbf{C}G$-resolution of the form

$$0 \longrightarrow \mathbf{C}G \longrightarrow \mathbf{C}G^d \longrightarrow \mathbf{C}G \longrightarrow \mathbf{C} \longrightarrow 0\,.$$

From this it is clear that the homology groups $H_n(G,\ell^p(G))$, $H^n(G,\ell^p(G))$, $\overline{H}_n(G,\ell^p(G))$ and $\overline{H}^n(G,\ell^p(G))$ are all zero for $n \geq 3$. Warren Dicks and I determined the ℓ^2-Betti numbers of such groups [2], Theorem 4.2; an immediate consequence of this is that $H_2(G,\ell^p(G)) = \overline{H}_2(G,\ell^p(G)) = 0$ for $p \leq 2$. The proof of this depended on the results that a torsion-free one-relator group is left orderable, and that if H is a left orderable group, $0 \neq \alpha \in \mathbf{C}H$ and $0 \neq \theta \in \ell^2(H)$, then $\alpha\theta \neq 0$ [4], Theorem 2. However if $p > 2$, then we can have $0 \neq \alpha \in \mathbf{C}H$ and $0 \neq \theta \in \ell^p(H)$ with $\alpha\theta = 0$; see [5] for information about this. Thus we have the following conjecture:

CONJECTURE 49.1. *Let G be a finitely generated torsion-free one-relator group and let $1 \leq p < \infty$. Then $H_2(G,\ell^p(G)) = \overline{H}_2(G,\ell^p(G)) = 0$.*

By Poincaré duality, this conjecture is true if G is a surface group, orientable or not. If Conjecture 49.1 is true, then it follows from Kappos's [3], Proposition 3.5 that $\overline{H}^2(G,\ell^p(G)) = 0$ for all $p > 1$. The situation for unreduced cohomology is less clear. Let us consider the special case $p = 2$. Recall that $\langle x, y \mid x^2y^2 = 1\rangle$ is the Klein bottle group. Our next conjecture is:

CONJECTURE 49.2. *Let G be a finitely generated torsion-free one-relator group. Then $H^2(G,\ell^2(G)) = 0$, provided G has neither $\mathbf{Z}\times\mathbf{Z}$ nor the Klein bottle group as a free factor.*

This conjecture is also true if G is a surface group, orientable or not. On the other hand if G is $\mathbf{Z}\times\mathbf{Z}$ or the Klein bottle group, then $H^2(G,\ell^2(G)) \neq 0$, and then it follows from the Mayer–Vietoris sequence that $H^2(G*\mathbf{Z},\ell^2(G*\mathbf{Z})) \neq 0$; of course $G*\mathbf{Z}$ is still a torsion-free one-relator group.

Finally we consider the first homology groups. A result of Guichardet [1], Theorem A shows that if G is an arbitrary infinite group, then the natural epimorphism $H^1(G,\ell^2(G)) \to \overline{H}^1(G,\ell^2(G))$ is an isomorphism if and only if G is non-amenable. This leads to the following conjecture.

CONJECTURE 49.3. *Let G be a finitely generated torsion-free one-relator group that has neither $\mathbf{Z} \times \mathbf{Z}$ nor the Klein bottle group as a free factor. Then the natural epimorphism $H_1(G, \ell^2(G)) \to \overline{H}_1(G, \ell^2(G))$ is an isomorphism.*

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