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FINITENESS PROPERTIES OF FUNCTION-FIELD-ARITHMETIC GROUPS

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Let K be a global function field, and S a finite nonempty set of pairwise inequivalent valuations on K. We let $\mathcal{O}_S \leq K$ be the corresponding ring of S-integers, and we let $\mathfrak{G}$ be a simple K-group. For any $v \in S$, we let K_v be the completion of K with respect to v.

The finiteness properties of function-field-arithmetic groups such as $\mathfrak{G}(\mathcal{O}_S)$ have been of interest for about 50 years. Some of that interest has fallen on which of these groups are of type FP_m for a given m. For example, for which m is it true that $\mathrm{SL}_n(\mathbf{F}_q[t])$ is of type FP_m where $\mathbf{F}_q[t]$ denotes a polynomial ring with one variable and coefficients in a finite field with q elements.

We recall that a group Γ is *of type* FP_m if there exists a partial resolution

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

of the trivial $\mathbf{Z}\Gamma$-module $\mathbf{Z}$ by finitely generated projective $\mathbf{Z}\Gamma$-modules.

All of the evidence thus far indicates the existence of a solution for the following

CONJECTURE 65.1. *With $\mathfrak{G}(\mathcal{O}_S)$ as above and with $k = \sum_{v\in S} \mathrm{rank}_{K_v}(\mathfrak{G})$, the arithmetic group $\mathfrak{G}(\mathcal{O}_S)$ is of type* FP_m *if and only if either* $\mathrm{rank}_K(\mathfrak{G}) = 0$ *or $k > m$.*

For example, according to the above conjecture $\mathrm{SL}_n(\mathbf{F}_q[t])$ should be of type $\mathrm{FP}_{(n-2)}$ but not of type $\mathrm{FP}_{(n-1)}$. In fact this was shown independently by Abels and Abramenko for large values of q.

Plenty of other evidence exists to support the conjecture in general including a proof of the "only if" implication [2]. The existing evidence for the conjecture can be found in papers of Abels, Abramenko, Behr, Bux–Wortman, Hurrelbrink, Keller, Kneser, Lubotzky, McHardy, Nagao, O'Meara, Rehmann–Soulé, Serre, Splitthoff, and Stuhler. For detailed references, see e.g. [1] and [2].

REFERENCES

[1] Behr, H. Arithmetic groups over function fields. I. A complete characterization of finitely generated and finitely presented arithmetic subgroups of reductive algebraic groups. *J. Reine Angew. Math. 495* (1998), 79–118.

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