

The regularity conjecture in the cohomology of groups

Autor(en): **Benson, Dave**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **54 (2008)**

Heft 1-2

PDF erstellt am: **25.05.2024**

Persistenter Link: <https://doi.org/10.5169/seals-109879>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

10

THE REGULARITY CONJECTURE IN THE COHOMOLOGY OF GROUPS

by Dave BENSON

Let k be a field of characteristic p and let G be a finite group.

CONJECTURE 10.1. *The Castelnuovo–Mumford regularity of the cohomology ring is equal to zero:*

$$\text{Reg } H^*(G, k) = 0.$$

This conjecture was first announced at the opening workshop of the MSRI commutative algebra year ([1]), as a refinement of a conjecture of Benson and Carlson ([4]). Subsequent work on the conjecture was reported in [2] and [3].

We begin with the definitions. Let H be a finitely generated graded commutative k -algebra, with $H^0 = k$ and $H^i = 0$ for $i < 0$ (e.g., $H = H^*(G, k)$). Write \mathfrak{m} for the maximal ideal generated by the elements of positive degree. If M is a graded H -module then the local cohomology is doubly graded: $H_{\mathfrak{m}}^{i,j} M$ denotes the part in local cohomological degree i and internal degree j . Local cohomology can either be regarded as the right derived functors of the \mathfrak{m} -torsion functor $\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \exists n \geq 0, \mathfrak{m}^n \cdot x = 0\}$, or as the cohomology of the stable Koszul complex (see for example Theorem 3.5.6 of Bruns and Herzog [6]).

The a -invariants of M are defined to be

$$a_{\mathfrak{m}}^i(M) = \max\{j \in \mathbf{Z} \mid H_{\mathfrak{m}}^{i,j} M \neq 0\}$$

(or $a_{\mathfrak{m}}^i(M) = -\infty$ if $H_{\mathfrak{m}}^i M = 0$).

The Castelnuovo–Mumford regularity of M is then defined as

$$\text{Reg } M = \max_{i \geq 0} \{a_{\mathfrak{m}}^i(M) + i\}.$$

Of particular interest is the regularity of the ring itself, $\text{Reg } H$.

The reason for the interest in local cohomology of group cohomology comes from the Greenlees version ([7]) of Benson–Carlson duality ([4]), in the form of a spectral sequence

$$H_{\mathfrak{m}}^{i,j} H^*(G, k) \Rightarrow H_{-i-j}(G, k).$$

In particular, the existence of the “last survivor” of [4] shows the following ([2]):

THEOREM 10.2. $\text{Reg } H^*(G, k) \geq 0$.

The regularity conjecture is known to hold in the following situations :

- $H^*(G, k)$ is Cohen–Macaulay; e.g., groups with abelian Sylow p -subgroups; groups with extraspecial Sylow 2-subgroups with $p = 2$; groups of Lie type with characteristic coprime to p ([1]).
- Krull dimension minus depth at most two; e.g., 2-groups of order ≤ 64 ([2]).
- Symmetric and alternating groups in any characteristic; these are examples where Krull dimension minus depth is arbitrarily large ([3]).

There is also a corresponding conjecture for compact Lie groups. Let G be a compact Lie group of dimension d , and suppose that the adjoint action of G on $\text{Lie}(G)$ preserves orientation. Then there is a spectral sequence (Benson–Greenlees [5])

$$H_{\mathfrak{m}}^{i,j} H^*(BG; k) \Rightarrow H_{-i-j-d}(BG; k).$$

CONJECTURE 10.3. $\text{Reg } H^*(BG; k) = -d$.

To explain the orientation condition, let $G = T^3 \rtimes \mathbf{Z}/2$, a semidirect product of a 3-torus by an involution acting through inversion, and k be a field of characteristic $\neq 2$. Then

$$H^*(BG; k) = H^*(BT; k)^{\mathbf{Z}/2}$$

is Cohen–Macaulay but not Gorenstein, and $\text{Reg } H^*(BG; k) = -5$. The appropriate modification in this situation is that if ε denotes the orientation character, then $\text{Reg } H^*(BG; \varepsilon) = -d$.

ADDED IN PROOF. David Green has verified the regularity conjecture for all groups of order 128. See D.J. GREEN. ‘Testing Benson’s regularity conjecture’. Preprint arXiv: math.GR/0710.2311 (2007).

REFERENCES

- [1] BENSON, D.J. Commutative algebra in the cohomology of groups. In: *Trends in Commutative Algebra*. MSRI Publications, vol. 51, Cambridge Univ. Press, 2004, 1–50.
- [2] —— Dickson invariants, regularity and computation in group cohomology. *Illinois J. Math.* 48 (2004), 171–197.
- [3] —— On the regularity conjecture for the cohomology of finite groups. (Preprint, 2005.) To appear in *Proc. Edinb. Math. Soc.*
- [4] BENSON, D.J. and J.F. CARLSON. Projective resolutions and Poincaré duality complexes. *Trans. Amer. Math. Soc.* 342 (1994), 447–488.
- [5] BENSON, D.J. and J.P.C. GREENLEES. Commutative algebra for cohomology rings of classifying spaces of compact Lie groups. *J. Pure Appl. Algebra* 122 (1997), 41–53.
- [6] BRUNS, W. and J. HERZOG. *Cohen–Macaulay Rings*. Cambridge Studies in Advanced Mathematics 39, Cambridge Univ. Press, Cambridge, 1993.
- [7] GREENLEES, J.P.C. Commutative algebra in group cohomology. *J. Pure Appl. Algebra* 98 (1995), 151–162.

Dave Benson

King’s College
Aberdeen AB24 3UE
Scotland
United Kingdom
e-mail: bensondj@maths.abdn.ac.uk