L'Enseignement Mathématique (2) 54 (2008), 29–31

10

THE REGULARITY CONJECTURE IN THE COHOMOLOGY OF GROUPS

by Dave Benson

Let k be a field of characteristic p and let G be a finite group.

CONJECTURE 10.1. *The Castelnuovo–Mumford regularity of the cohomology ring is equal to zero:*

$$\operatorname{Reg} H^*(G,k) = 0\,.$$

This conjecture was first announced at the opening workshop of the MSRI commutative algebra year ([1]), as a refinement of a conjecture of Benson and Carlson ([4]). Subsequent work on the conjecture was reported in [2] and [3].

We begin with the definitions. Let H be a finitely generated graded commutative k-algebra, with $H^0 = k$ and $H^i = 0$ for $i < 0$ (e.g., $H = H^*(G,k)$). Write $\mathfrak{m}$ for the maximal ideal generated by the elements of positive degree. If M is a graded H-module then the local cohomology is doubly graded: $H^{i,j}_{\mathfrak{m}} M$ denotes the part in local cohomological degree i and internal degree j. Local cohomology can either be regarded as the right derived functors of the $\mathfrak{m}$-torsion functor $\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \exists n \geq 0,\, \mathfrak{m}^n.x = 0\}$, or as the cohomology of the stable Koszul complex (see for example Theorem 3.5.6 of Bruns and Herzog [6]).

The *a-invariants* of M are defined to be

$$a^i_{\mathfrak{m}}(M) = \max\{j \in \mathbf{Z} \mid H^{i,j}_{\mathfrak{m}} M \neq 0\}$$

(or $a^i_{\mathfrak{m}}(M) = -\infty$ if $H^i_{\mathfrak{m}} M = 0$).

The *Castelnuovo–Mumford regularity* of M is then defined as

$$\operatorname{Reg} M = \max_{i \geq 0}\{a^i_{\mathfrak{m}}(M) + i\}\,.$$

Of particular interest is the regularity of the ring itself, Reg H.

The reason for the interest in local cohomology of group cohomology comes from the Greenlees version ([7]) of Benson–Carlson duality ([4]), in the form of a spectral sequence

$$H^{i,j}_{\mathfrak{m}} H^*(G,k) \Rightarrow H_{-i-j}(G,k)\,.$$

In particular, the existence of the "last survivor" of [4] shows the following ([2]):

THEOREM 10.2. Reg $H^*(G,k) \geq 0$.

The regularity conjecture is known to hold in the following situations:

- $H^*(G,k)$ is Cohen–Macaulay; e.g., groups with abelian Sylow p-subgroups; groups with extraspecial Sylow 2-subgroups with $p=2$; groups of Lie type with characteristic coprime to p ([1]).
- Krull dimension minus depth at most two; e.g., 2-groups of order ≤ 64 ([2]).
- Symmetric and alternating groups in any characteristic; these are examples where Krull dimension minus depth is arbitrarily large ([3]).

There is also a corresponding conjecture for compact Lie groups. Let G be a compact Lie group of dimension d, and suppose that the adjoint action of G on Lie(G) preserves orientation. Then there is a spectral sequence (Benson–Greenlees [5])

$$H^{i,j}_{\mathfrak{m}} H^*(BG;k) \Rightarrow H_{-i-j-d}(BG;k)\,.$$

CONJECTURE 10.3. Reg $H^*(BG;k) = -d$.

To explain the orientation condition, let $G = T^3 \rtimes \mathbf{Z}/2$, a semidirect product of a 3-torus by an involution acting through inversion, and k be a field of characteristic $\neq 2$. Then

$$H^*(BG;k) = H^*(BT;k)^{\mathbf{Z}/2}$$

is Cohen–Macaulay but not Gorenstein, and Reg $H^*(BG;k) = -5$. The appropriate modification in this situation is that if ε denotes the orientation character, then Reg $H^*(BG;\varepsilon) = -d$.

ADDED IN PROOF. David Green has verified the regularity conjecture for all groups of order 128. See D. J. GREEN. 'Testing Benson's regularity conjecture'. Preprint arXiv: math.GR/0710.2311 (2007).

REFERENCES

[1] BENSON, D. J. Commutative algebra in the cohomology of groups. In: *Trends in Commutative Algebra*. MSRI Publications, vol. 51, Cambridge Univ. Press, 2004, 1–50.

[2] —— Dickson invariants, regularity and computation in group cohomology. *Illinois J. Math. 48* (2004), 171–197.

[3] —— On the regularity conjecture for the cohomology of finite groups. (Preprint, 2005.) To appear in *Proc. Edinb. Math. Soc.*

[4] BENSON, D. J. and J. F. CARLSON. Projective resolutions and Poincaré duality complexes. *Trans. Amer. Math. Soc. 342* (1994), 447–488.

[5] BENSON, D. J. and J. P. C. GREENLEES. Commutative algebra for cohomology rings of classifying spaces of compact Lie groups. *J. Pure Appl. Algebra 122* (1997), 41–53.

[6] BRUNS, W. and J. HERZOG. *Cohen–Macaulay Rings*. Cambridge Studies in Advanced Mathematics *39*, Cambridge Univ. Press, Cambridge, 1993.

[7] GREENLEES, J. P. C. Commutative algebra in group cohomology. *J. Pure Appl. Algebra 98* (1995), 151–162.

Dave Benson

King's College
Aberdeen AB24 3UE
Scotland
United Kingdom
e-mail: bensondj@maths.abdn.ac.uk