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## INFINITE TOPOLOGY OF CURVE COMPLEXES AND NON-POINCARÉ DUALITY OF TEICHMÜLLER MODULAR GROUPS

by Nikolai IVANOV and Lizhen JI<sup>\*</sup>)

ABSTRACT. In this note, we fill a gap in the literature by proving that the Teichmüller modular groups (mapping class groups) are not virtual Poincaré duality groups and that the complexes of curves of surfaces have infinite homotopy type (i.e. are not homotopy equivalent to a finite CW-complex).

### 1. INTRODUCTION

Let  $S$  be an orientable surface. Let  $\text{Diff}^\pm(S)$  be the group of all diffeomorphisms of  $S$ , and let  $\text{Diff}^0(S)$  be its identity component. Then  $\text{Mod}_S^\pm = \text{Diff}(S)/\text{Diff}^0(S)$  is called the *extended mapping class group* or the *extended Teichmüller modular group* of  $S$ . Let  $\text{Diff}(S)$  be the subgroup of orientation preserving diffeomorphisms of  $S$ . Then  $\text{Mod}_S = \text{Diff}(S)/\text{Diff}^0(S)$  is called the *mapping class group* or the *Teichmüller modular group* of  $S$ .

When  $S$  is a closed surface of genus 1, for example, when  $S = \mathbf{Z}^2 \backslash \mathbf{R}^2$  is the standard torus, then  $\text{Mod}_S^\pm$  can be identified with  $GL(2, \mathbf{Z})$ , and  $\text{Mod}_S$  can be identified with  $SL(2, \mathbf{Z})$ . If  $S$  is a closed surface of genus  $g \geq 2$ , or more generally an oriented surface of negative Euler characteristic  $\chi(S)$ , then  $\text{Mod}_S^\pm$  and  $\text{Mod}_S$  can be considered as natural generalizations of  $GL(2, \mathbf{Z})$  and  $SL(2, \mathbf{Z})$  respectively.

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The group  $SL(2, \mathbf{Z})$  is an important example of arithmetic subgroups of semisimple linear algebraic groups; in particular, it is the first group in the classical family of arithmetic subgroups  $SL(n, \mathbf{Z})$ ,  $n \geq 2$  of  $SL(n, \mathbf{R})$ . Arithmetic subgroups  $\Gamma$  of semisimple linear algebraic groups  $\mathbf{G}$  (defined over  $\mathbf{Q}$ ) enjoy many good properties and are special among discrete groups. For example, they are finitely presented and enjoy other finiteness properties such as being of type  $FP_\infty$  and of type  $FL$ . Borel and Serre showed in [BoS] that arithmetic subgroups  $\Gamma$  are virtual duality groups and that their virtual cohomological dimension can be computed explicitly (we outline parts of this theory below). The cohomology groups of a natural family of arithmetic subgroups such as  $SL(n, \mathbf{Z})$  stabilize as  $n \rightarrow \infty$ . See [Se] and [Bo1] for a summary and [Bo2] for a computation of the stable real cohomology groups.

Let  $G = \mathbf{G}(\mathbf{R})$  be the real locus of  $\mathbf{G}$ , let  $K \subset G$  be a maximal compact subgroup, and let  $X = G/K$  be the associated symmetric space. Assume that  $\Gamma$  is a torsion-free arithmetic subgroup of  $\mathbf{G}(\mathbf{Q})$ . Then the locally symmetric space  $\Gamma \backslash X$  is an aspherical manifold with fundamental group  $\Gamma$ . If  $\Gamma$  is a cocompact subgroup of  $G$ , then the Poincaré duality for  $\Gamma \backslash X$  implies that  $\Gamma$  is a Poincaré duality group. On the other hand, if the arithmetic subgroup  $\Gamma$  is not a cocompact discrete subgroup of  $G$  (as is  $SL(n, \mathbf{Z})$  or rather its finite index torsion-free subgroups, for example), then [BoS] (as was noted by Serre; see [BiE], p.124, *Note Added in Proof*) implies that  $\Gamma$  is *not a Poincaré duality group*, in particular, it cannot be realized as the fundamental group of a closed aspherical manifold. Instead of being a Poincaré duality group,  $\Gamma$  enjoys a weaker property, being a *duality group* in the sense of Bieri-Eckmann [BiE]. In Section 2, we will recall the definition of duality groups and of Poincaré duality groups (see, in particular, the formulas (2.1), (2.2), (2.3)).

In the context of Teichmüller modular groups, the *Teichmüller space*  $T_S$  of  $S$  plays the role of the symmetric space  $X$ , and the canonical action of  $Mod_S$  on  $T_S$  plays the role of the action of  $\Gamma$  on  $X$ . This analogy was discovered by Harvey [Harv1], [Harv2] who, in particular, was motivated by the problem of providing analogues of some constructions of Borel-Serre [BoS].

In a series of remarkable papers [Ha3]–[Ha1] (see also [Iv1] for another approach to these and other related results, and [Ha4], [Iv5] for expository accounts), Harer, motivated by the analogy between the groups  $Mod_S$  and the arithmetic groups (which was also originally pointed out by Harvey [Harv1], [Harv2]) established many properties of  $Mod_S$  similar to those of arithmetic subgroups.

For simplicity, in the sequel we assume that  $S$  is a closed orientable surface of genus  $g \geq 2$ , and  $Mod_S$  will be also denoted by  $Mod_g$ . All results are also true for surfaces with non-empty boundary, with minor modifications.

We are especially interested in the following result (see [Ha3], Theorem 4.1, and [Iv1], Theorem 6.6 for a different proof).

**THEOREM 1.1.** *For every torsion-free subgroup  $\Gamma \subset Mod_g$  of finite index,  $\Gamma$  is a duality group (in the sense of Bieri-Eckmann [BiE]) of dimension  $4g - 5$ . Therefore,  $Mod_g$  is a virtual duality group of virtual cohomological dimension  $4g - 5$ .*

This theorem is an analogue of some results of Borel-Serre [BoS]. The crucial ingredient of the proof of these results of Borel-Serre is the Solomon-Tits theorem about the homotopy type of the spherical Tits building  $\Delta(\mathbf{G})$ , which (as proved by Borel-Serre) is homotopy equivalent to the boundary of the partial Borel-Serre compactification of  $X$  constructed in [BoS]. An analogue of the Borel-Serre compactification for the Teichmüller spaces was suggested by Harvey [Harv2], as also an analogue of the Tits building  $\Delta(\mathbf{G})$ . The latter analogue is the *complex of curves*  $\mathcal{C}(S)$  of  $S$ , which plays a fundamental role in the theory of Teichmüller modular groups  $Mod_S$  and, in particular, in the present paper. See the beginning of Section 2 for the definition of  $\mathcal{C}(S)$ .

The underlying topological space of the spherical Tits building  $\Delta(\mathbf{G})$  consists of a *non-disjoint*, countably infinite union of spheres with one sphere for every apartment. Hence it is not surprising that  $\Delta(\mathbf{G})$  is homotopy equivalent to a countably infinite bouquet of spheres, as is indeed the case by the Solomon-Tits Theorem. Basically, this is proved by showing that the intersection of any two distinct apartments is contractible (see [Br1], p.92, Theorem 2, and [Ku]). On the other hand, the curve complex  $\mathcal{C}(S)$  does not have any structure of apartments, and its homotopy type is not clear (or easy to guess) from the definition.

In [Ha3] (see also [Ha4], Chap. 4, § 1, and [Iv1], § 3 or [Iv5], Theorem 3.3.A for a different proof), Harer proved the following analogue of the Solomon-Tits Theorem.

**THEOREM 1.2.** *The curve complex  $\mathcal{C}(S)$  is homotopy equivalent to a bouquet of spheres  $\bigvee S^n$ , where the dimension  $n = 2g - 2$ .*

The proof of this theorem given in [Ha3], [Ha4] proceeds in two steps: (1)  $\mathcal{C}(S)$  is  $(n-1)$ -connected; (2) the homotopy dimension of  $\mathcal{C}(S)$  is bounded

from above by  $n$ . Therefore,  $\mathcal{C}(S)$  is homotopy equivalent to a bouquet  $\bigvee S^n$ . The alternative proof of [Iv5], §3.3, p. 546, [Iv1], §3 follows the same general outline. These arguments leave open the possibility that the number of spheres in the above bouquet is equal to 0, and  $\mathcal{C}(S)$  is contractible. In fact, this is not the case, as the following theorem shows.

**THEOREM 1.3.** *The curve complex  $\mathcal{C}(S)$  is not contractible, and hence the bouquet of spheres  $\bigvee S^n$  in Theorem 1.2 contains at least one sphere.*

Motivated by Theorems 1.2, 1.3, and by the Solomon-Tits Theorem, it is also natural to expect the following improvement of Theorem 1.2.

**THEOREM 1.4.** *The curve complex  $\mathcal{C}(S)$  is homotopy equivalent to a countably infinite bouquet of spheres  $S^n$ , where  $n = 2g - 2$ .*

The algebraic counterpart of Theorems 1.3, 1.4 is the following result.

**THEOREM 1.5.** *For every torsion-free subgroup  $\Gamma \subset \text{Mod}_S$  of finite index,  $\Gamma$  is not a Poincaré duality group. Equivalently,  $\text{Mod}_S$  is not a virtual Poincaré duality group. In particular, no subgroup  $\Gamma$  of finite index of  $\text{Mod}_S$  can be realized as the fundamental group of a closed aspherical manifold.*

If we take seriously the analogy between the modular groups  $\text{Mod}_S$  and the arithmetic groups, and, in particular, the analogy between the Teichmüller space  $T_S$  of  $S$  and the symmetric spaces  $X = G/K$ , this result was to be expected. Namely, the arithmetic groups  $\Gamma$  are virtual Poincaré duality groups if and only if they act on  $X$  cocompactly, and the action of  $\text{Mod}_S$  on  $T_S$  is not cocompact, as is well known. This heuristic argument is based only on the fact that the complex of curves  $\mathcal{C}(S)$  is non-empty (this is equivalent to the action on  $T_S$  being non-cocompact). But the actual proof proceeds differently. Namely, the proof of Theorem 1.4 is based on Theorem 1.5, the proof of which is, in turn, based on properties of the Mess subgroups  $[M]$  of  $\text{Mod}_S$ . The proof of Theorem 1.3 is more direct, but also uses properties of the group  $\text{Mod}_S$ ; namely, the fact that  $\text{Mod}_S$  has non-zero (virtual) cohomological dimension.

Theorem 1.3, while undoubtedly known to people working in this field (including J. Harer and the first author), was for a long time not stated or proved explicitly in the literature. At the same time it was used in several places, for example, in understanding relations between the automorphism

group of the curve complex  $\mathcal{C}(S)$  and the mapping class group of  $S$  (see [Iv4], [Kor1], [Kor2], and [Luo]), and was also used implicitly in the proof of Theorem 1.1. We have included the relevant part of the proof of Theorem 1.1 at the end of Section 3.

While Theorem 1.3 is at least implicitly contained in the papers [Ha3], [Iv1], Theorems 1.4 and 1.5 are not discussed there, and the proof of these theorems which we offer below is based on later results due to Mess [M]. One can safely say that everybody assumed that these results were true, but nobody cared to write down a proof. (After a preliminary version of this note was written up, T. Farrell pointed out that Harer claimed in [Ha4], p. 180, lines 20–22, that the dualizing module  $I$  (denoted by  $C$  in this note) for a torsion-free subgroup  $\Gamma$  of  $\text{Mod}_S$  of finite index has infinite rank, which is equivalent to Theorem 1.4 of this note.)

This question on the nontrivial topology of  $\mathcal{C}(S)$  has been raised by several persons and has caused some confusion. The goal of this note is to provide proofs of Theorems 1.3, 1.4 and 1.5.

The proofs are based on a consideration of cohomology groups of torsion-free subgroups of  $\text{Mod}_S$  of finite index and on the relation of the curve complex  $\mathcal{C}(S)$  to the boundary structure of a Borel-Serre type partial compactification  $\overline{T}_S^{BS}$  discovered by Harvey [Harv2] (or a truncation  $T_S(\varepsilon)$ ) of the Teichmüller space  $T_S$  of  $S$ . These ideas form the foundation of the theory presented in [Ha3], [Ha4], and [Iv1]. In particular, to a large extent our proof of Theorem 1.3 is already contained in these papers.

In order to have a fuller analogy between the arithmetic and the Teichmüller modular cases, it seems desirable to have a proof of Theorem 1.3 using the internal structure of  $\mathcal{C}(S)$  as in the case of the Solomon-Tits theorem for spherical Tits buildings. Note that cohomology of arithmetic groups and the action of arithmetic subgroups on the spherical Tits buildings  $\Delta(\mathbf{G})$  are not used to prove the Solomon-Tits Theorem. So the proof of Theorem 1.3 presented below looks like putting the cart before the horse. It is also worthwhile to understand if there are some special subcomplexes of  $\mathcal{C}(S)$  which play the role of apartments in Tits buildings.

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## 2. PRELIMINARIES

For simplicity, we assume that  $S$  is a closed oriented surface of genus  $g \geq 2$ . By the definition, the vertices of the curve complex  $\mathcal{C}(S)$  are free homotopy classes  $\langle c \rangle$  of simple closed curves  $c$  in  $S$ . Vertices  $\langle c_1 \rangle, \dots, \langle c_{k+1} \rangle$  form the vertices of a  $k$ -simplex if and only if they are all different and every two curves  $c_{i_1}$  and  $c_{i_2}$  for  $1 \leq i_1 < i_2 \leq k+1$  are isotopic to disjoint curves. It is well known and easy to see that  $\mathcal{C}(S)$  is a simplicial complex of dimension  $3g-4$ .

Let  $T_S$ , which is also denoted by  $T_g$ , be the *Teichmüller space* of hyperbolic metrics on  $S$ . The curve complex  $\mathcal{C}(S)$  was first introduced by Harvey [Harv2] in order to construct a partial compactification  $\overline{T}_g^{BS}$  of  $T_g$  (his project was completed by the first author in [Iv2]), which is similar to and motivated by the Borel-Serre partial compactification  $\overline{X}^{BS}$  of symmetric spaces  $X = G/K$ . The latter is a manifold with corners whose boundary faces (or components) are parametrized by simplexes of a spherical Tits building associated with  $G$ .

More specifically, if  $G = \mathbf{G}(\mathbf{R})$  is the real locus of a linear semisimple algebraic group  $\mathbf{G}$  defined over  $\mathbf{Q}$ , then the boundary of  $\overline{X}^{BS}$  is decomposed into boundary components  $e(\mathbf{P})$  parametrized by  $\mathbf{Q}$ -parabolic subgroups  $\mathbf{P}$  of  $\mathbf{G}$  satisfying the following two conditions:

1. The closure of every  $e(\mathbf{P})$  in  $\overline{X}^{BS}$  is contractible.
2.  $e(\mathbf{P}_1)$  is contained in the closure of  $e(\mathbf{P}_2)$  if and only if  $\mathbf{P}_1$  is contained in  $\mathbf{P}_2$ .

Recall that the *spherical Tits building*  $\Delta(\mathbf{G})$  of an algebraic group  $\mathbf{G}$  defined over  $\mathbf{Q}$  is a simplicial complex with one simplex  $\sigma_{\mathbf{P}}$  for each proper parabolic subgroup  $\mathbf{P}$  (defined over  $\mathbf{Q}$ ) of  $\mathbf{G}$  satisfying the following two conditions:

1. Vertices of  $\Delta(\mathbf{G})$  correspond to maximal proper  $\mathbf{Q}$ -parabolic subgroups of  $\mathbf{G}$ .
2.  $\sigma_{\mathbf{P}_1}$  is contained in  $\sigma_{\mathbf{P}_2}$  if and only if  $\mathbf{P}_2$  is contained in  $\mathbf{P}_1$ . In particular, maximal proper  $\mathbf{Q}$ -parabolic subgroups  $\mathbf{P}_1, \dots, \mathbf{P}_{k+1}$  form the vertices of a  $k$ -simplex if and only if their intersection  $\mathbf{P}_1 \cap \dots \cap \mathbf{P}_{k+1}$  is a parabolic subgroup of  $\mathbf{G}$ .

The boundary components  $e(\mathbf{P})$  of  $\bar{X}^{BS}$  correspond to the simplexes of the Tits building  $\Delta(\mathbf{G})$  of  $\mathbf{G}$ . Furthermore, the partial Borel-Serre compactification  $\bar{X}^{BS}$  is a real analytic manifold with corners whose boundary  $\partial\bar{X}^{BS}$  has the same homotopy type as the (geometric realization of the) building  $\Delta(\mathbf{G})$ .

For every torsion-free arithmetic subgroup  $\Gamma \subset \mathbf{G}(\mathbf{Q})$ , the action of  $\Gamma$  on  $X$  extends to a proper and continuous action on  $\bar{X}^{BS}$  such that the quotient  $\Gamma \backslash \bar{X}^{BS}$  is a compact manifold with corners. In particular, it is a finite  $B\Gamma$ -space, i.e., a classifying space  $B\Gamma$  which is a finite CW-complex. These results are used in an essential way in studying cohomology groups of  $\Gamma$ . In fact, it is shown in [BoS] (see also [BiE]) that every arithmetic subgroup  $\Gamma$  of  $\mathbf{G}(\mathbf{Q})$  is a virtual duality group with virtual cohomological dimension equal to  $\dim X - r$ , where  $r$  is equal to the  $\mathbf{Q}$ -rank of  $\mathbf{G}$ . This result motivated the work [Ha3] and other related results on mapping class groups (see [Ha4] for a summary).

Note that the dimension of  $\Delta(\mathbf{G})$  is equal to  $r - 1$ . It is a well-known theorem of Solomon-Tits (see [Br1], p. 93, Theorem 2) that  $\Delta(\mathbf{G})$  is homotopy equivalent to a countably infinite bouquet of spheres of dimension  $r - 1$ . In fact, if we fix a chamber  $C$  of the building  $\Delta(\mathbf{G})$ , there is one sphere for each apartment of the building  $\Delta(\mathbf{G})$  containing  $C$ . This implies that when the  $\mathbf{Q}$ -rank  $r$  is positive, every arithmetic subgroup of  $\mathbf{G}(\mathbf{Q})$  is not a Poincaré duality group, in particular, cannot be realized as the fundamental group of a closed aspherical manifold. (This was noted by Serre; see [BiE], p. 124, *Note Added in Proof*.) Note also that the condition that  $r > 0$  is equivalent to the condition that every arithmetic subgroup  $\Gamma$  of  $\mathbf{G}(\mathbf{Q})$  is not a cocompact discrete subgroup of  $G$ .

For the convenience of the reader, we recall (see, for example [BiE]) that a discrete group  $\Gamma$  is called a *Poincaré duality group* of dimension  $n$  if there exists a right  $\Gamma$ -module structure on  $\mathbf{Z}$  such that

$$(2.1) \quad H^k(\Gamma, A) \cong H_{n-k}(\Gamma, \mathbf{Z} \otimes A)$$

for all  $k$  and all  $\Gamma$ -modules  $A$ . In this case,

$$(2.2) \quad H^n(\Gamma, \mathbf{Z}\Gamma) \cong \mathbf{Z}.$$



If  $\Gamma$  admits a classifying space which is a *closed* (i.e. compact without boundary) manifold  $M$  or equivalently if  $\Gamma$  is the fundamental group of a closed aspherical manifold  $M$ , then the Poincaré duality for  $M$  implies that  $\Gamma$  is a Poincaré duality group.

A discrete group  $\Gamma$  is called a *duality group* of dimension  $n$  (in the sense of Bieri-Eckmann [BiE]) with respect to a right  $\Gamma$ -module  $C$ , called the *dualizing module*, if there is an element  $e \in H_n(\Gamma, C)$  such that the cap-product with  $e$  induces isomorphisms

$$(2.3) \quad H^k(\Gamma, A) \cong H_{n-k}(\Gamma, C \otimes A)$$

for all  $k$  and all left  $\Gamma$ -modules  $A$ . In this case, the *cohomological dimension* of  $\Gamma$  is equal to  $n$ .

Let us turn to the Teichmüller case. The Borel-Serre partial compactification  $\overline{T}_g^{BS}$  of [Harv2] is also a manifold with corners with boundary faces  $B_\sigma$  parametrized by simplexes  $\sigma$  of the curve complex  $\mathcal{C}(S)$  such that  $B_{\sigma_1}$  is contained in the closure of  $B_{\sigma_2}$  if and only if  $\sigma_2$  is contained in  $\sigma_1$ . In this sense, the curve complex  $\mathcal{C}(S)$  is similar to spherical Tits buildings  $\Delta(\mathbf{G})$ . The construction of  $\overline{T}_g^{BS}$  was not worked out in detail in [Harv2] and  $\overline{T}_g^{BS}$  was later constructed in [Iv2].

A version of  $\overline{T}_g^{BS}$  which is easier to construct and to deal with is provided by a *truncation*  $T_g(\varepsilon)$  of the Teichmüller space  $T_g$ . It is due to the first author [Iv3] (see also [Iv5], Theorem 5.4.A). For every point  $x = (S, d) \in T_g$ , where  $d$  is a hyperbolic metric on  $S$ , denote the length of a closed geodesic  $c$  in  $S$  with respect to the hyperbolic metric  $d$  by  $\ell_x(c)$ . For  $\varepsilon$  sufficiently small, define

$$T_g(\varepsilon) = \{x = (S, d) \in T_g \mid \text{for every closed geodesic } c \subset S, \ell_x(c) \geq \varepsilon\}.$$

The smallness of  $\varepsilon$  is required so that two geodesics of lengths less than  $\varepsilon$  do not intersect. Such  $\varepsilon$  always exists due to the collar theorem (see [Bu], §4.1 for example).

Let  $Mod_g = Mod_S$  be the Teichmüller modular group of  $S$ . Then  $Mod_g$  acts properly on  $T_g$ . Since points of every orbit of  $Mod_g$  in  $T_g$  represent isometric metrics on  $S$ , it is clear that  $Mod_g$  leaves  $T_g(\varepsilon)$  invariant. It follows from the Mumford compactness criterion that the quotient  $Mod_g \backslash T_g(\varepsilon)$  is compact.

The following result (see [Iv5], Theorem 5.4.A) is crucial for us.

PROPOSITION 2.1. *The space  $T_g(\varepsilon)$  is a contractible manifold with corners. Its boundary faces when  $T_g(\varepsilon)$  is considered as a manifold with corners are contractible and parametrized by the simplexes of the curve complex  $\mathcal{C}(S)$ , and the whole boundary  $\partial T_g(\varepsilon)$  is homotopy equivalent to  $\mathcal{C}(S)$ . In particular, for every torsion-free subgroup  $\Gamma \subset \text{Mod}_g$  of finite index, the quotient  $\Gamma \backslash T_g(\varepsilon)$  is a finite  $B\Gamma$ -space.*

The crucial part of the proof of this theorem is to show that  $T_g(\varepsilon)$  is contractible, and this follows from the fact  $T_g(\varepsilon)$  is a deformation retract of  $T_g$ . The correspondence between its boundary components and the simplexes of  $\mathcal{C}(S)$  is described as follows. Each simplex  $\sigma$  of  $\mathcal{C}(S)$  gives a collection of disjoint and non-isotopic simple closed geodesics. The corresponding boundary face of  $\partial T_g(\varepsilon)$  consists of those marked hyperbolic metrics on  $S$  where the geodesics in this collection have length exactly equal to  $\varepsilon$ . Therefore, each boundary face of  $\partial T_g(\varepsilon)$  is basically the product of a truncated Teichmüller space and an Euclidean space (corresponding to the Fenchel-Nielsen twist parameters), and hence also contractible.

### 3. PROOFS OF THEOREMS 1.3, 1.4 AND 1.5

We start with the proof of Theorem 1.3. It is similar to the arguments in [Ha3], [Ha4], [Iv1], [Iv5], and, of course, to the arguments in [BoS] where Borel and Serre prove that arithmetic subgroups of linear semisimple algebraic groups are virtual duality groups.

*Proof of Theorem 1.3.* Let  $\Gamma \subset \text{Mod}_g$  be a torsion-free subgroup of finite index. The idea of the proof is to show that if  $\mathcal{C}(S)$  is homotopy trivial, i.e., if there is no  $S^n$  in the bouquet  $\bigvee S^n$  of spheres in Theorem 1.2, then the cohomological dimension of  $\Gamma$ , denoted by  $\text{cd}\Gamma$ , is equal to zero. But this is impossible since  $\Gamma$  is a torsion-free infinite group.

By Proposition 2.1,  $\Gamma \backslash T_g(\varepsilon)$  is a compact  $B\Gamma$ -space. Hence by [Iv5], Corollary 6.1.G,  $\text{cd}\Gamma$  is equal to the maximum number  $n$  such that

$$H_c^n(T_g(\varepsilon), \mathbf{Z}) \neq 0,$$

where  $H_c^n(\cdot)$  denotes the cohomology with compact supports. Denote the dimension of  $T_g(\varepsilon)$  by  $d$ . Following [Iv5], Theorem 6.1.H (or [BoS]), we see that by the Poincaré-Lefschetz duality,

$$(3.1) \quad H_c^n(T_g(\varepsilon), \mathbf{Z}) = H_{d-n}(T_g(\varepsilon), \partial T_g(\varepsilon); \mathbf{Z}).$$

Since  $T_g(\varepsilon)$  is contractible, the homology long exact sequence gives

$$(3.2) \quad H_{d-n}(T_g(\varepsilon), \partial T_g(\varepsilon); \mathbf{Z}) \cong \tilde{H}_{d-n-1}(\partial T_g(\varepsilon), \mathbf{Z}).$$

By Proposition 2.1,  $\partial T_g(\varepsilon)$  is homotopy equivalent to  $\mathcal{C}(S)$ . Therefore,

$$(3.3) \quad H_c^n(T_g(\varepsilon), \mathbf{Z}) \cong \tilde{H}_{d-n-1}(\mathcal{C}(S)).$$

Suppose that  $\mathcal{C}(S)$  is homotopy equivalent to a point. Then equation (3.3) implies that for every  $n \geq 1$ ,

$$H_c^n(T_g(\varepsilon), \mathbf{Z}) = 0.$$

This in turn implies that  $\text{cd } \Gamma = 0$ .

It is well known that for every pair of groups  $\Gamma' \subset \Gamma$ , we have  $\text{cd } \Gamma' \leq \text{cd } \Gamma$ . (See [Br1], Proposition 2.4, p. 187 or [Iv5], §6.4, p. 584.) Since  $\Gamma$  is torsion-free and infinite, it contains  $\mathbf{Z}$  as a subgroup. A classifying space of  $\mathbf{Z}$  is given by the circle  $S^1$ , and hence clearly  $\text{cd } \mathbf{Z} = 1$ . This implies that  $\text{cd } \Gamma \geq 1$ . This contradiction implies that the homotopy type of  $\mathcal{C}(S)$  cannot be trivial. The proof of Theorem 1.3 is complete.  $\square$

REMARK 3.1. The proof of Theorem 1.3 can be arranged in slightly different ways. By Theorem 1.2 and the arguments based on equations (3.1), (3.2) and (3.3) (and taking into account that  $H^\nu(\Gamma, \mathbf{Z}\Gamma) \cong H_c^\nu(T_g(\varepsilon), \mathbf{Z})$ ; see [Iv5], Lemma 6.1.F), we see that for  $i \neq \nu = 4g - 5$ ,  $H^i(\Gamma, \mathbf{Z}\Gamma) = 0$ . By Proposition 2.1 (see Remark 3.2 below),  $\Gamma$  is of type *FP*. If  $H^\nu(\Gamma, \mathbf{Z}\Gamma) = 0$ , then [Br1], Proposition 6.7 (see p. 202) implies that the cohomological dimension  $\text{cd } \Gamma$  is equal to 0.

Since  $\Gamma$  is torsion-free and contains  $\mathbf{Z}$ , the last paragraph of the above proof of Theorem 1.3 implies  $\text{cd } \Gamma \geq 1$ . This, in turn, implies that  $H^\nu(\Gamma, \mathbf{Z}\Gamma) \neq 0$ . By the above equations again, it follows that the bouquet  $\bigvee S^n$  for  $\mathcal{C}(S)$  contains at least one sphere.

Or, alternatively, by [Br1], Theorem 10.1 (see p. 220), and its proof, this non-vanishing result implies that  $\Gamma$  is a duality group of dimension  $\nu$ . Since the lower dimensional cohomology groups of  $\Gamma$  are known to be non-zero (see [Ha1] for example), the dualizing module  $\tilde{H}_n(\mathcal{C}(S), \mathbf{Z})$  is not 0, which implies that  $\mathcal{C}(S)$  is not contractible.

The last approach has the disadvantage of being dependent on very difficult calculations of Harer [Ha1].

REMARK 3.2. We also note that although it is not stated explicitly in [Br2], Chapter VIII, Theorem 10.1 (c), under the assumption that  $\Gamma$  is a torsion-free infinite group of type  $FP$ , the non-vanishing of  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  is implied by other conditions:  $H^i(\Gamma, \mathbf{Z}\Gamma) = 0$  for  $i \neq \nu$ . In fact, since  $\Gamma$  contains  $\mathbf{Z}$ , it follows from the second paragraph of the previous remark that the group  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  must be non-zero.

This non-vanishing of  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  is used at several places in the book of Brown [Br2]. For example, to show that the cohomological dimension of  $\Gamma$  is equal to  $n$  in [Br2], Chapter VIII, Proposition 6.7, in the proof (c) implies (d) in [Br2], Chapter VIII, Theorem 10.1, and also in [Br2], Exercise 5 (b) in Section VIII.6.

Note that if  $\Gamma$  is a torsion-free subgroup of finite index in  $Mod_g$ , then the condition that  $\Gamma$  is of type  $FP$  follows from the fact that  $\Gamma$  acts freely on the manifold with corners  $T_g(\varepsilon)$  with a compact quotient  $\Gamma \backslash T_g(\varepsilon)$  together with the fact that a compact manifold with corners admits a finite triangulation, which implies that  $\Gamma$  admits a finite  $B\Gamma$ -space and hence is of type  $FL$ , which is stronger than being of type  $FP$ .

*Proof of Theorem 1.5.* Suppose that  $\Gamma$  is a Poincaré duality group. Then, by a theorem of R. Strebel (see [St], 1.2), for every subgroup  $\Gamma' \subset \Gamma$  of infinite index,  $\text{cd } \Gamma' < \text{cd } \Gamma$ .

Therefore, in order to prove Theorem 1.5, it suffices to find a subgroup  $\Gamma'$  of  $\Gamma$  of infinite index (i.e.  $[\Gamma : \Gamma'] = +\infty$ ) with  $\text{cd } \Gamma' = \text{cd } \Gamma$ . One can take a Mess subgroup  $B_g$  [M] as such a group. See [Iv5], §6.3 for an accessible exposition of the relevant part of the Mess work. The Mess subgroups  $B_g$  are constructed by an induction on  $g$ . The construction involves a lot of arbitrary choices, so actually there are (infinitely) many Mess subgroups of  $Mod_g$  for any given  $g$ . The last step of the construction of  $B_g$  is to take a subgroup of  $Mod_g$  supported on a subsurface of genus  $g-1$  with one boundary component (it is constructed from a  $B_{g-1}$ ) and to add to it a Dehn twist  $t_c$  about a nontrivial circle  $c$  disjoint from this subsurface. Clearly, the subgroup  $B_g$  is contained in the centralizer of  $t_c$ . In particular, it does not contain nontrivial powers of Dehn twists about circles not isotopic to a circle disjoint from  $c$ . It follows that  $B_g$  is of infinite index in  $Mod_g$ . The intersection  $\Gamma' = B_g \cap \Gamma$  is of infinite index in  $\Gamma$ . A crucial result about the Mess subgroups  $B_g$  is that  $B_g$  is the fundamental group of a closed topological aspherical manifold of dimension equal to the  $\text{cd } \Gamma = 4g - 5$ ; see [Iv5], §6.3, Theorem 6.3.A. In particular,  $\text{cd } \Gamma' = \text{cd } B_g = \text{cd } \Gamma$ . This completes the proof that  $\Gamma$  is not a Poincaré duality group.  $\square$

*Proof of Theorem 1.4.* Suppose that  $\mathcal{C}(S)$  is homotopy equivalent to a bouquet of finitely many spheres  $S^n$ .

Let  $\nu = 4g - 5$  be the virtual cohomological dimension of  $Mod_g$ . If  $d = \dim T_g(\varepsilon) = 6g - 6$ , then  $d - \nu - 1 = 6g - 6 - (4g - 5) - 1 = 2g - 2 = n$ . For any torsion-free subgroup  $\Gamma$  of  $Mod_g$  of finite index, we have by [Iv5], Lemma 6.1.F,

$$H^\nu(\Gamma, \mathbf{Z}\Gamma) \cong H_c^\nu(T_g(\varepsilon), \mathbf{Z}).$$

Further, by the equations (3.1), (3.2) and (3.3) above (and the fact that  $d - \nu - 1 = n$ ), we have

$$(3.4) \quad H^\nu(\Gamma, \mathbf{Z}\Gamma) \cong H_c^\nu(T_g(\varepsilon), \mathbf{Z}) \cong \tilde{H}_n(\partial T_g(\varepsilon), \mathbf{Z}) \cong \tilde{H}_n(\mathcal{C}(S), \mathbf{Z})$$

is a finitely generated abelian group.

By a theorem of Farrell (see [Fa], Theorem 3), if  $\Gamma$  is a finitely presented group of type *FP*, is a duality group of dimension  $\nu$ , and if  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  is finitely generated, then  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  is cyclic and hence  $\Gamma$  is a Poincaré duality group. By Theorem 1.1, a torsion-free subgroup  $\Gamma$  of  $Mod_g$  of finite index is a duality group. By Remark 3.2, such a subgroup  $\Gamma$  is of type *FP*. This implies that  $\Gamma$  is a Poincaré duality group. But this contradicts Theorem 1.5.

It follows that  $\mathcal{C}(S)$  is homotopy equivalent to a bouquet of infinitely many spheres  $S^n$ . Since  $\mathcal{C}(S)$  is a countable simplicial complex, there are only countably many spheres in the bouquet  $\bigvee S^n$ .  $\square$

REMARK 3.3. One can avoid using [Fa], Theorem 3 in the proof of Theorem 1.4 by applying [Br1], Exercise 4 in Section VIII.6. According to this exercise, if  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  is finitely generated (where  $\nu = \text{cd } \Gamma$ ), then for every subgroup  $\Gamma' \subset \Gamma$  of infinite index,  $\text{cd } \Gamma' < \text{cd } \Gamma$ . Notice that, if there are only finitely many spheres in the bouquet, then  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  is finitely generated. Using the Mess subgroups as in the proof of Theorem 1.5, one can complete the proof.

REMARK 3.4. In order to prove Theorem 1.4, i.e. that the bouquet  $\bigvee S^n$  contains infinitely many spheres, it would be sufficient to show that the bouquet  $\bigvee S^n$  contains at least two spheres. In fact, by equation (3.4), in this case  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  contains a subgroup isomorphic to  $\mathbf{Z}^2$ . Since for the Poincaré duality groups the equation (2.2) holds, this implies that  $\Gamma$  is not a Poincaré duality group, and then one would be able to complete the proof by appealing to Farrell's theorem again. Unfortunately, it seems that there is no direct proof of the fact that the bouquet  $\bigvee S^n$  contains at least two spheres. Compare Remark 3.5.

*Sketch of a proof of Theorem 1.1.* For the sake of completeness and clarification of any possible confusion, we outline a proof of Theorem 1.1. The arguments are the same as in [Ha3], [Ha4], [Iv1], and [BoS] (see [BoS], Theorem 8.6.5), and prove that  $Mod_g$  is a virtual duality group of dimension  $\nu = (6g - 6) - (2g - 2) - 1 = 4g - 5$ .

Specifically, let  $\Gamma$  be a torsion-free subgroup of  $Mod_g$  of finite index. By [BiE], Theorem 2.3 (see also [Br2], Theorem 10.1, p.220), it suffices to prove the following claim:

(3.5)

$H^n(\Gamma, \mathbf{Z}\Gamma) = 0$  for  $n \neq \nu$ , and  $H^\nu(\Gamma, \mathbf{Z}\Gamma)$  is *nonzero* and torsion-free.

Since  $\Gamma \backslash T_g(\varepsilon)$  is a finite  $B\Gamma$ -space, by [Iv5], Lemma 6.1.F, we have

$$(3.6) \quad H^n(\Gamma, \mathbf{Z}\Gamma) = H^n(\Gamma \backslash T_g(\varepsilon), \mathbf{Z}\Gamma) \cong H_c^n(T_g(\varepsilon), \mathbf{Z}).$$

Since the boundary  $\partial T_g(\varepsilon)$  is homotopy equivalent to a nonempty bouquet of spheres  $\bigvee S^n$ , where  $n = 2g - 2$ , the same arguments as in the proof of Theorem 1.3 above (see equations (3.1), (3.2) and (3.3)) prove (3.5). Therefore,  $\Gamma$  is a duality group and hence  $Mod_g$  is a virtual duality group.

**REMARK 3.5.** The analogy between the arithmetic and the Teichmüller cases breaks down in one important respect. Recall that the topological dimension of  $\mathcal{C}(S)$  is equal to  $3g - 4$ , since there are at most  $3g - 3$  disjoint non-homotopic simple closed curves in the surface  $S$  of genus  $g$ . It is not clear why, in contrast to the Solomon-Tits theorem for spherical Tits buildings, the topological dimension  $3g - 4$  of  $\mathcal{C}(S)$  is much higher than its homotopy dimension  $2g - 2$ . This is quite different from spherical Tits buildings whose topological and homotopy dimensions agree.

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