

# Bounds for cohomology classes

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# 18

## BOUNDS FOR COHOMOLOGY CLASSES

by Marc BURGER, Alessandra Iozzi, Nicolas MONOD and Anna WIENHARD

Let  $G$  be a simple Lie group (connected and with finite centre). Consider the continuous cohomology  $H^*(G, \mathbf{R})$  of  $G$ , which can be defined for instance with the familiar bar-resolutions of the Eilenberg–MacLane cohomology, except that the cochains are required to be *continuous* maps on  $G$  (or equivalently smooth or just measurable).

**CONJECTURE 18.1.** *Every cohomology class of  $H^*(G, \mathbf{R})$  is bounded, i.e. is represented by a bounded cocycle.*

Recall that  $H^*(G, \mathbf{R})$  is isomorphic to the algebra of invariant differential forms on the symmetric space associated to  $G$ , hence to a relative cohomology of Lie algebras and thus moreover to the cohomology of the *compact dual space* associated to  $G$ . It is however not understood how these isomorphisms interact with boundedness of cochains (compare Dupont [6]).

We emphasise also that, unlike for discrete groups,  $H^*(G, \mathbf{R})$  does not coincide with the cohomology of the classifying space  $BG$ . There is however a natural transformation  $H^*(BG, \mathbf{R}) \rightarrow H^*(G, \mathbf{R})$  and we refer to its image as the *primary characteristic classes*. By a difficult result of M. Gromov [7], the latter are indeed bounded; M. Bucher-Karlsson gave a simpler proof of this fact in her thesis [1].

In order to prove the above conjecture, it would suffice to establish the boundedness of the *secondary invariants* of Cheeger–Simons; indeed, Dupont–Kamber proved that the latter together with the primary classes generate  $H^*(G, \mathbf{R})$  as an algebra.

An important example where boundedness was established very recently is the class of the *volume form* of the associated symmetric space. Using estimates by Connell–Farb [5], Lafont–Schmidt [8] provided bounded cocycles

in all cases except  $\mathrm{SL}_3(\mathbf{R})$ , the latter case being settled by M. Bucher-Karlsson [2] (a previous proof of R. Savage [11] is incorrect). It follows that the fundamental class of closed locally symmetric spaces is bounded; as explained by M. Gromov, this provides a non-zero lower bound for the *minimal volume* of such a manifold, i.e. a non-trivial lower bound for its volume with respect to *any* (suitably normalised) Riemannian metric.

Many more questions are related to the above conjecture via the following steps listed in increasing order of refinement: (i) find a bounded cocycle representing a given class; (ii) establish a sharp numerical bound for that class; (iii) determine the equivalence class of the cocycle up to boundaries of bounded cochains only.

The latter point leads one to introduce the (continuous) *bounded cohomology*  $H_b^*$  of groups or spaces, where all cochains are required to be bounded. There is then an obvious natural transformation

$$(*) \quad H_b^*(-, \mathbf{R}) \longrightarrow H^*(-, \mathbf{R})$$

and the above conjecture amounts to the surjectivity of that map for a connected simple Lie group with finite centre. As of now, there is not a single simple Lie group for which  $H_b^*(G, \mathbf{R})$  is known; all the partial results are however consistent with a positive answer to the following:

QUESTION 18.2. *Is the map  $(*)$  an isomorphism?*

For instance, the answer is yes in degree two [3] (and trivially yes in degrees 0, 1); for  $G = \mathrm{SL}_n(\mathbf{R})$ , it is also yes in degree three (see [4] for  $n = 2$  and [10] for  $n \geq 3$ ).

The functor  $H_b^*$  is quite interesting for discrete groups as well and has found applications notably to representation theory, dynamics, geometry and ergodic theory. This notwithstanding, *there is not a single countable group for which  $H_b^*(-, \mathbf{R})$  is known*, unless it is known to vanish in all degrees (e.g. for amenable groups). In any case, the map  $(*)$  fails dramatically either to be injective or surjective in many examples. Most known results regard the degree two, with for instance a large supply of groups having an infinite-dimensional  $H_b^2(-, \mathbf{R})$ , including the non-Abelian free group  $F_2$ . Interestingly, the surjectivity of the map  $(*)$  (with more general coefficients) in degree two *characterises non-elementary Gromov-hyperbolic groups* (Mineyev [9]).

It appears that new techniques are required in higher degrees. Here is a test on which to try them:

QUESTION 18.3. *For which degrees  $n$  is  $H_b^n(F_2, \mathbf{R})$  non-trivial?*

It is known to be non-trivial for  $n = 2, 3$ . (Triviality for  $n = 1$  and non-triviality for  $n = 0$  are elementary to check for any group.)

#### REFERENCES

- [1] BUCHER-KARLSSON, M. Characteristic classes and bounded cohomology. Ph.D. thesis, ETHZ Dissertation Nr. 15636, 2004.
- [2] —— Simplicial volume of locally symmetric spaces covered by  $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}(3)$ . *Geom. Dedicata* 125 (2007), 203–224.
- [3] BURGER, M. and N. MONOD. Continuous bounded cohomology and applications to rigidity theory (with an appendix by M. Burger and A. Iozzi). *Geom. Funct. Anal.* 12 (2002), 219–280.
- [4] BURGER, M. and N. MONOD. On and around the bounded cohomology of  $\mathrm{SL}_2$ . In: *Rigidity in Dynamics and Geometry (Cambridge, 2000)*, 19–37. Springer, Berlin, 2002.
- [5] CONNELL, C. and B. FARB. The degree theorem in higher rank. *J. Differential Geom.* 65 (2003), 19–59.
- [6] DUPONT, J. L. Bounds for characteristic numbers of flat bundles. In: *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978)*, 109–119. Springer, Berlin, 1979.
- [7] GROMOV, M. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.* 56 (1982), 5–99.
- [8] LAFONT, J.-F. and B. SCHMIDT. Simplicial volume of closed locally symmetric spaces of non-compact type. *Acta Math.* 197 (2006), 129–143.
- [9] MINEYEV, I. Bounded cohomology characterizes hyperbolic groups. *Quart. J. Math.* 53 (2002), 59–73.
- [10] MONOD, N. Stabilization for  $\mathrm{SL}_n$  in bounded cohomology. In: *Discrete Geometric Analysis*. Contemp. Math. 347, 191–202. Amer. Math. Soc., Providence, RI, 2004.
- [11] SAVAGE, R. P. JR. The space of positive definite matrices and Gromov’s invariant. *Trans. Amer. Math. Soc.* 274 (1982), 239–263.

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