Cohomological finiteness conditions : spaces versus H-spaces

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COHOMOLOGICAL FINITENESS CONDITIONS: SPACES VERSUS H-SPACES

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We wish to ask a very naive and classically flavored question. Consider a finite complex X and an integer n. Does its n-connected cover $X\langle n\rangle$ satisfy any cohomological finiteness property? When X is an H-space we have:

THEOREM 22.1 ([3]). *Let X be a finite H-space and n an integer. Then $H^*(X\langle n\rangle;\mathbf{F}_p)$ is finitely generated as an algebra over the Steenrod algebra.*

This leads naturally to ask whether the same statement holds for arbitrary spaces. Of course, some restriction on the fundamental group will be needed, as the universal cover of $S^1 \vee S^2$ is an infinite wedge of copies of S^2.

QUESTION 22.2. *Let X be a simply connected finite space and $n \geq 2$. Is $H^*(X\langle n\rangle;\mathbf{F}_p)$ finitely generated as an algebra over the Steenrod algebra?*

The "difference" between a space and its n-connected cover is a Postnikov piece. Thus, a first step towards a solution to Question 22.2 would be to understand the cohomology of finite type Postnikov pieces.

QUESTION 22.3. *Is the cohomology of a finite-type Postnikov piece finitely generated as an algebra over the Steenrod algebra?*

Again, we know from [3], Corollary 3.8, that the answer is yes if the Postnikov piece is an H-space. The proof of Theorem 22.1 is based on L. Smith's analysis of the Eilenberg–Moore spectral sequence, and the following algebraic result, whose proof relies deeply on the Borel–Hopf structure theorem.

THEOREM 22.4 ([3]). *Let A be an unstable Hopf algebra which is finitely generated as an algebra over the Steenrod algebra. Then so is any unstable Hopf subalgebra $B \subset A$.*

For plain unstable algebras, this is false, as pointed out to us by Hans-Werner Henn. Consider the unstable algebra $H^*(\mathbf{C}P^\infty \times S^2; \mathbf{F}_p) \cong \mathbf{F}_p[x] \otimes E(y)$ where both x and y have degree 2. Turn the ideal generated by y into an unstable subalgebra by adding 1. This is isomorphic, as an unstable algebra, to $\mathbf{F}_p \oplus \Sigma^2\mathbf{F}_p \oplus \Sigma^2\widetilde{H}^*(\mathbf{C}P^\infty; \mathbf{F}_p)$, which is not finitely generated.

Where do these questions come from? The condition that $H^*(X; \mathbf{F}_p)$ is finitely generated as an algebra over the Steenrod algebra is equivalent to the condition that the indecomposables $QH^*(X; \mathbf{F}_p)$ are finitely generated as a module over the Steenrod algebra. This guarantees that $QH^*(X; \mathbf{F}_p)$ lives in the Krull filtration of the category $\mathcal{U}$ of unstable modules, introduced by Schwartz in [5]: an unstable module M lives in $\mathcal{U}_n$ if and only if $\overline{T}^{n+1}M = 0$, where $\overline{T}$ denotes Lannes' reduced T functor. This algebraic filtration can be compared with Bousfield's $B\mathbf{Z}/p$-nullification filtration, [1] (a connected space X is $B\mathbf{Z}/p$-null if the space of pointed maps $\mathrm{map}_*(B\mathbf{Z}/p, X)$ is contractible).

THEOREM 22.5 ([2]). *Let X be a connected H-space satisfying that $T_V H^*(X; \mathbf{F}_p)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X; \mathbf{F}_p)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1}X$ is $B\mathbf{Z}/p$-null.*

Dwyer and Wilkerson have shown in [4] that the case $n = 0$ holds for arbitrary spaces. However, our methods rely so deeply on the H-structure that we still don't know if one should look for a positive or negative answer to our last question.

QUESTION 22.6. *Let X be a connected space such that $T_V H^*(X; \mathbf{F}_p)$ is of finite type for any elementary abelian p-group V, and let $n \geq 1$. Is it true that $QH^*(X; \mathbf{F}_p)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1}X$ is $B\mathbf{Z}/p$-null?*

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