

# Ternary cubic forms and étale algebras

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Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **55 (2009)**

Heft 1-2

PDF erstellt am: **25.05.2024**

Persistenter Link: <https://doi.org/10.5169/seals-110099>

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## TERNARY CUBIC FORMS AND ÉTALE ALGEBRAS

by Mélanie RACZEK and Jean-Pierre TIGNOL<sup>\*</sup>)

The configuration of inflection points on a nonsingular cubic curve in the complex projective plane has a well-known remarkable feature: a line through any two of the nine inflection points passes through a third inflection point. Therefore the inflection points and the 12 lines through them form a tactical configuration  $(9_4, 12_3)$ , which is the configuration of points and lines of the affine plane over the field with 3 elements ([3, p. 295], [7, p. 242]). This property was used by Hesse to show that the inflection points of a ternary cubic over the rationals are defined over a solvable extension, see [11, § 110]. As a result, any ternary cubic can be brought to a normal form  $x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3$  over a solvable extension of the base field<sup>1)</sup>. The purpose of this paper is to investigate this extension.

Throughout the paper, we denote by  $F$  an arbitrary field of characteristic different from 3, by  $F_s$  a separable closure of  $F$  and by  $\Gamma = \text{Gal}(F_s/F)$  its Galois group. Let  $V$  be a 3-dimensional  $F$ -vector space and let  $f \in S^3(V^*)$  be a cubic form on  $V$ . Assume that  $f$  has no singular zero in the projective plane  $\mathbf{P}_V(F_s)$ . Then the set  $\mathfrak{I}(f) \subseteq \mathbf{P}_V(F_s)$  of inflection points has 9 elements. There are 12 lines in  $\mathbf{P}_V(F_s)$  that contain three points of  $\mathfrak{I}(f)$ ; they are called *inflectional lines*. Their set  $\mathfrak{L}(f)$  is partitioned into four 3-element subsets  $\mathfrak{T}_0, \mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$  called *inflectional triangles*, which have the property that each inflection point is incident to exactly one line of each triangle. Let  $\mathfrak{T}(f) = \{\mathfrak{T}_0, \mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$ . There is a canonical map  $\mathfrak{L}(f) \rightarrow \mathfrak{T}(f)$ , which carries every inflectional line to the unique triangle that contains it. The Galois

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<sup>\*</sup>) The second author is partially supported by the Fund for Scientific Research F.R.S.–FNRS (Belgium).

<sup>1)</sup> We are grateful to the erudite anonymous referee who pointed out that the normal form of cubics was obtained by Hesse in [5, § 20, Aufgabe 2] *before* he proved (in [6]) that the equation of inflection points is solvable by radicals.

group  $\Gamma$  acts on  $\mathfrak{I}(f)$ , hence also on  $\mathfrak{L}(f)$  and  $\mathfrak{T}(f)$ , and the canonical map  $\mathfrak{L}(f) \rightarrow \mathfrak{T}(f)$  is a triple covering of  $\Gamma$ -sets, in the terminology of [9, §2.2]. Galois theory associates to the  $\Gamma$ -set  $\mathfrak{L}(f)$  a 12-dimensional étale  $F$ -algebra  $L(f)$ , which is a cubic étale extension of the 4-dimensional étale  $F$ -algebra  $T(f)$  associated to  $\mathfrak{T}(f)$ . We show in §4 that if one of the inflectional triangles, say  $\mathfrak{T}_0$ , is defined over  $F$ , hence preserved under the  $\Gamma$ -action, then there are decompositions

$$T(f) \simeq F \times N, \quad L(f) \simeq K \times M,$$

where  $N$  and  $K$  are cubic étale  $F$ -algebras whose corresponding  $\Gamma$ -sets are  $\mathfrak{X}(N) = \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$  and  $\mathfrak{X}(K) = \mathfrak{T}_0$  respectively, and  $M$  is a 9-dimensional étale  $F$ -algebra containing  $N$ , associated to  $K$  and a unit  $a \in K^\times$ . One can then identify the vector space  $V$  with  $K$  in such a way that

$$(0.1) \quad f(X) = T_K(a^{-1}X^3) - 3\lambda N_K(X) \quad \text{for some } \lambda \in F,$$

where  $T_K$  and  $N_K$  are the trace and the norm of the  $F$ -algebra  $K$ . Conversely, if  $f$  can be reduced to the form (0.1), then one of the inflectional triangles is defined over  $F$ , and  $\mathfrak{X}(K)$  is isomorphic to the set of lines of the triangle. Note that the (*generalized*) *Hesse normal form*

$$a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 - 3\lambda x_1 x_2 x_3$$

is the particular case of (0.1) where  $K = F \times F \times F$  (i.e., the  $\Gamma$ -action on  $\mathfrak{X}(K)$  is trivial) and  $a = (a_1^{-1}, a_2^{-1}, a_3^{-1})$ . As an application, we show that the form  $T_K(X^3)$  can be reduced over  $F$  to a generalized Hesse normal form if and only if  $K$  has the form  $F[\sqrt[3]{d}]$  for some  $d \in F^\times$ , see Example 4.4.

The 9-dimensional étale  $F$ -algebra  $M$  associated to a cubic étale  $F$ -algebra  $K$  and a unit  $a \in K^\times$  was first defined by Markus Rost in relation with Morley's theorem. We are grateful to Markus for allowing us to quote from his private notes [10] in §2.

For background information on cubic curves, we refer to [3], Chapter 11 of [7], or [2].

## 1. ÉTALE ALGEBRAS OVER A FIELD

An *étale  $F$ -algebra* is a finite-dimensional commutative  $F$ -algebra  $A$  such that  $A \otimes_F F_s \simeq F_s \times \cdots \times F_s$ ; see [1, Ch. 5, §6] or [8, §18] for various other characterizations of étale  $F$ -algebras. For any étale  $F$ -algebra  $A$ , we denote by  $\mathfrak{X}(A)$  the set of  $F$ -algebra homomorphisms  $A \rightarrow F_s$ . This is a finite set with

cardinality  $|\mathfrak{X}(A)| = \dim_F A$ . Composition with automorphisms of  $F_s$  endows  $\mathfrak{X}(A)$  with a  $\Gamma$ -set structure, and  $\mathfrak{X}$  is a contravariant functor that defines an anti-equivalence of categories between the category  $\text{Et}_F$  of étale  $F$ -algebras and the category  $\text{Set}_\Gamma$  of finite  $\Gamma$ -sets, see [1, Ch. 5, §10] or [8, (18.4)].

Let  $G$  be a finite group of automorphisms of an étale  $F$ -algebra  $A$ . The group  $G$  acts faithfully on the  $\Gamma$ -set  $\mathfrak{X}(A)$ .

PROPOSITION 1.1. *If  $G$  acts freely (i.e., without fixed points) on  $\mathfrak{X}(A)$ , then*

$$H^1(G, A^\times) = 1.$$

*Proof.* The  $G$ -action on  $\mathfrak{X}(A)$  maps each  $\Gamma$ -orbit on a  $\Gamma$ -orbit, since the actions of  $G$  and  $\Gamma$  commute. We may thus decompose  $\mathfrak{X}(A)$  into a disjoint union

$$\mathfrak{X}(A) = \mathfrak{X}_1 \coprod \dots \coprod \mathfrak{X}_n,$$

where each  $\mathfrak{X}_i$  is a union of  $\Gamma$ -orbits permuted by  $G$ . Using the anti-equivalence between  $\text{Et}_F$  and  $\text{Set}_\Gamma$ , we obtain a corresponding decomposition of  $A$  into a direct product of étale  $F$ -algebras

$$A = A_1 \times \dots \times A_n.$$

The  $G$ -action preserves each  $A_i$ , hence

$$H^1(G, A^\times) = H^1(G, A_1^\times) \times \dots \times H^1(G, A_n^\times).$$

It therefore suffices to prove that  $H^1(G, A^\times) = 1$  when  $G$  acts transitively on the  $\Gamma$ -orbits in  $\mathfrak{X}(A)$ . These  $\Gamma$ -orbits are in one-to-one correspondence with the primitive idempotents of  $A$ . Let  $e$  be one of these idempotents and let  $H \subseteq G$  be the subgroup of automorphisms that leave  $e$  fixed. Let also  $B = eA$ . The map  $g \otimes b \mapsto g(b)$  for  $g \in G$  and  $b \in B$  induces isomorphisms of  $G$ -modules

$$A = \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} B, \quad A^\times = \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} B^\times,$$

hence the Eckmann–Faddeev–Shapiro lemma (see for instance [4, Prop. (6.2), p. 73]) yields an isomorphism

$$H^1(G, A^\times) \simeq H^1(H, B^\times).$$

Now,  $B$  is a field and each element  $h \in H$  restricts to an automorphism of  $B$ . Let  $\xi \in \mathfrak{X}(A)$  be such that  $\xi(e) = 1$ , hence  $\xi(x) = \xi(ex)$  for all  $x \in A$ . If  $h \in H$  restricts to the identity on  $B$  then

$$eh(x) = h(ex) = ex \quad \text{for all } x \in A,$$

and hence

$$\xi(h(x)) = \xi(x) \quad \text{for all } x \in A.$$

It follows that  $h$  leaves  $\xi$  fixed, hence  $h = 1$  since  $G$  acts freely on  $\mathfrak{X}(A)$ . Therefore  $H$  embeds injectively in the group of automorphisms of  $B$ . Hilbert's Theorem 90 then yields  $H^1(H, B^\times) = 1$ , see [8, (29.2)].

## 2. MORLEY ALGEBRAS

Let  $K$  be an étale  $F$ -algebra of dimension 3. To every unit  $a \in K^\times$  we associate an étale  $F$ -algebra  $M(K, a)$  of dimension 9 by a construction due to Markus Rost [10], which will be crucial for the description of the  $\Gamma$ -action on inflectional lines of a nonsingular cubic, see Theorem 3.2.

DEFINITION 2.1. Let  $D$  be the *discriminant algebra* of  $K$  (see [8, p. 291]); this is a 2-dimensional étale  $F$ -algebra such that  $K \otimes_F D$  is the  $S_3$ -Galois closure of  $K$ , see [8, § 18.C]. We thus have  $F$ -algebra automorphisms  $\sigma, \rho$  of  $K \otimes_F D$  such that

$$\sigma|_D = \text{Id}_D, \quad \rho|_K = \text{Id}_K, \quad \sigma^3 = \rho^2 = \text{Id}_{K \otimes_F D}, \quad \text{and} \quad \rho\sigma = \sigma^2\rho.$$

We identify each element  $x \in K$  with its image  $x \otimes 1$  in  $K \otimes_F D$  and denote its norm by  $N_K(x)$ .

Now, fix an element  $a \in K^\times$ . Let  $s, t$  be indeterminates and consider the quotient  $F$ -algebra

$$A = K \otimes_F D[s, t] / (s^3 - \sigma^2(a)\sigma(a)^{-1}, t^3 - N_K(a)).$$

Since the characteristic is different from 3, every  $F$ -algebra homomorphism  $K \otimes_F D \rightarrow F_s$  extends in 9 different ways to  $A$ , so  $A$  is an étale  $F$ -algebra. Abusing notation, we also denote by  $s$  and  $t$  the images in  $A$  of the indeterminates. Straightforward computations show that  $\sigma$  and  $\rho$  extend to automorphisms of  $A$  by letting

$$\sigma(s) = s t \sigma^2(a)^{-1}, \quad \sigma(t) = t, \quad \rho(s) = s^{-1}, \quad \rho(t) = t,$$

and that the extended  $\sigma, \rho$  satisfy  $\sigma^3 = \rho^2 = \text{Id}_A$  and  $\rho\sigma = \sigma^2\rho$ , so they generate a group  $G$  of automorphisms of  $A$  isomorphic to the symmetric group  $S_3$ . The subalgebra of  $A$  fixed under  $G$  is called the *Morley  $F$ -algebra* associated with  $K$  and  $a$ . It is denoted by  $M(K, a)$ .

Since  $G$  acts freely on  $\mathfrak{X}(K \otimes_F D)$ , it also acts freely on  $\mathfrak{X}(A)$ , hence

$$\dim_F M(K, a) = 9.$$

It readily follows from the definition that  $M(K, a)$  contains the 3-dimensional étale  $F$ -algebra

$$N(K, a) = F[t], \quad \text{with } t^3 = N_K(a).$$

Clearly, if  $a' = \lambda k^3 a$  for some  $\lambda \in F^\times$  and  $k \in K^\times$ , then there is an isomorphism  $M(K, a') \simeq M(K, a)$  induced by  $s' \mapsto s\sigma^2(k)\sigma(k)^{-1}$ ,  $t' \mapsto t\lambda N_K(k)$ .

EXAMPLE 2.2. Let  $K = F \times F \times F$  and  $a = (a_1, a_2, a_3) \in K^\times$ . Then  $D \simeq F \times F$ , so  $K \otimes_F D \simeq F^6$ . We index the primitive idempotents of  $K \otimes D$  by the elements in  $G$ , so that the  $G$ -action on the primitive idempotents  $(e_\tau)_{\tau \in G}$  is given by

$$\theta(e_\tau) = e_{\theta\tau} \quad \text{for } \theta, \tau \in G.$$

We identify  $K$  with a subalgebra of  $K \otimes D$  by

$$(x_1, x_2, x_3) = x_1(e_{\text{Id}} + e_\rho) + x_2(e_\sigma + e_{\rho\sigma}) + x_3(e_{\sigma^2} + e_{\rho\sigma^2})$$

for  $x_1, x_2, x_3 \in F$ . Then  $A \simeq F^6[s, t]$  where

$$s^3 = \frac{\sigma^2(a)}{\sigma(a)} = \frac{a_2}{a_3}e_{\text{Id}} + \frac{a_3}{a_1}e_\sigma + \frac{a_1}{a_2}e_{\sigma^2} + \frac{a_3}{a_2}e_\rho + \frac{a_2}{a_1}e_{\sigma\rho} + \frac{a_1}{a_3}e_{\sigma^2\rho}$$

and

$$t^3 = a_1 a_2 a_3.$$

Let  $r = \sum_{\tau \in G} \tau(s) e_\tau \in M(K, a)$ . Then  $r^3 = \frac{a_2}{a_3}$  and  $M(K, a) = F[r, t]$ . Note that  $\left(\frac{r^2 t}{a_2}\right)^3 = \frac{a_1}{a_3}$ , so

$$M(K, a) \simeq F\left[\sqrt[3]{\frac{a_1}{a_3}}, \sqrt[3]{\frac{a_2}{a_3}}\right] \quad \text{and} \quad N(K, a) \simeq F[\sqrt[3]{a_1 a_2 a_3}].$$

EXAMPLE 2.3. Let  $K$  be an arbitrary cubic étale  $F$ -algebra and let  $a = 1$ . Let  $F[\omega]$  be the quadratic étale  $F$ -algebra with  $\omega^2 + \omega + 1 = 0$ . By the Chinese Remainder Theorem we have

$$N(K, 1) = F[t]/(t^3 - 1) \simeq F \times F[\omega].$$

The corresponding orthogonal idempotents in  $N(K, 1)$  are

$$e_1 = \frac{1}{3}(1 + t + t^2) \quad \text{and} \quad e_2 = \frac{1}{3}(2 - t - t^2).$$

Let  $A_1 = e_1A$  and  $A_2 = e_2A$ , so  $A = A_1 \oplus A_2$  and the  $G$ -action preserves  $A_1$  and  $A_2$ . Let

$$\begin{aligned} e_{11} &= \frac{1}{3}(1 + s + s^2)e_1 \in A_1, & e_{12} &= \frac{1}{3}(2 - s - s^2)e_1 \in A_1, \\ \varepsilon_1 &= \frac{1}{3}(1 + s + s^2)e_2 \in A_2, & \varepsilon_2 &= \frac{1}{3}(1 + st + s^2t^2)e_2 \in A_2, \\ \varepsilon_3 &= \frac{1}{3}(1 + st^2 + s^2t)e_2 \in A_2. \end{aligned}$$

These elements are pairwise orthogonal idempotents, and we have

$$e_1 = e_{11} + e_{12}, \quad e_2 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

The  $G$ -action fixes  $e_{11}$  and  $e_{12}$ , while

$$\begin{aligned} \sigma(\varepsilon_1) &= \varepsilon_2, & \sigma(\varepsilon_2) &= \varepsilon_3, & \sigma(\varepsilon_3) &= \varepsilon_1, \\ \rho(\varepsilon_1) &= \varepsilon_1, & \rho(\varepsilon_2) &= \varepsilon_3, & \rho(\varepsilon_3) &= \varepsilon_2. \end{aligned}$$

We have  $e_1t = e_1$  and  $e_{11}s = e_{11}$ , hence  $e_{11}A \simeq K \otimes D$  and  $e_{11}M(K, 1) \simeq F$ . On the other hand,  $e_{12}s$  is a primitive cube root of unity in  $e_{12}M(K, 1)$ . It is fixed under  $\sigma$  and  $\rho(e_{12}s) = e_{12}s^{-1}$ . Therefore we have

$$e_{12}A \simeq K \otimes D \otimes F[\omega] \quad \text{and} \quad e_{12}M(K, 1) \simeq (D \otimes F[\omega])^\rho,$$

where  $\rho$  acts non-trivially on  $D$  and  $F[\omega]$ . The quadratic étale algebra  $(D \otimes F[\omega])^\rho$  is the *composite* of  $D$  and  $F[\omega]$  in the group of quadratic étale  $F$ -algebras, see [9, Prop. 3.11]. It is denoted by  $D * F[\omega]$ . Finally, we have an isomorphism  $K \otimes F[\omega] \simeq e_2M(K, 1)$  by mapping  $x \in K$  to  $x\varepsilon_1 + \sigma(x)\varepsilon_2 + \sigma^2(x)\varepsilon_3$  and  $\omega$  to  $e_2t$ , so

$$M(K, 1) \simeq F \times (D * F[\omega]) \times (K \otimes F[\omega]).$$

Under this isomorphism, the inclusion  $N(K, 1) \hookrightarrow M(K, 1)$  is the map

$$F \times F[\omega] \rightarrow F \times (D * F[\omega]) \times (K \otimes F[\omega]), \quad (x, y) \mapsto (x, x, y).$$

In particular, if  $F$  contains a cube root of unity, then  $F[\omega] \simeq F \times F$  and

$$M(K, 1) \simeq F \times D \times K \times K.$$

The inclusion  $N(K, 1) \hookrightarrow M(K, 1)$  is then given by

$$F \times F \times F \rightarrow F \times D \times K \times K, \quad (x, y, z) \mapsto (x, x, y, z).$$

Details are left to the reader.

In the rest of this section, we show how the  $\Gamma$ -set  $\mathfrak{X}(M(K, a))$  can be characterized as the fibre of a certain (ramified) covering of the projective plane.

Viewing  $K$  as an  $F$ -vector space, we may consider the projective plane  $\mathbf{P}_K$ , whose points over the separable closure  $F_s$  are

$$\mathbf{P}_K(F_s) = \{x \cdot F_s^\times \mid x \in K \otimes_F F_s, x \neq 0\}.$$

Let

$$(2.1) \quad \pi: \mathbf{P}_K(F_s) \rightarrow \mathbf{P}_K(F_s), \quad x \cdot F_s^\times \mapsto x^3 \cdot F_s^\times \quad \text{for } x \in K \otimes_F F_s, x \neq 0.$$

We show in Theorem 2.6 below that there is an isomorphism of  $\Gamma$ -sets

$$\mathfrak{X}(M(K, a)) \simeq \pi^{-1}(a \cdot F_s^\times) \quad \text{for } a \in K^\times.$$

In view of the anti-equivalence between  $\text{Et}_F$  and  $\text{Set}_\Gamma$ , this result characterizes the Morley algebra  $M(K, a)$  up to isomorphism.

Until the end of this section, we fix  $a \in K^\times$  and denote  $M(K, a)$  simply by  $M$ . We identify  $K \otimes M$  with the subalgebra of  $A$  fixed under  $\rho$ .

LEMMA 2.4. *There exists  $u \in (K \otimes M)^\times$  such that  $s = \sigma^2(u) \sigma(u)^{-1}$ .*

*Proof.* Define a map  $c: G \rightarrow A^\times$  by

$$c(\text{Id}) = c(\sigma^2 \rho) = 1, \quad c(\sigma) = c(\rho) = s, \quad c(\sigma^2) = c(\sigma \rho) = \sigma^2(s)^{-1}.$$

Computation shows that  $s\sigma(s)\sigma^2(s) = 1$ , and it follows that  $c$  is a 1-cocycle. Proposition 1.1 yields an element  $v \in A^\times$  such that  $c(\tau) = v\tau(v)^{-1}$  for all  $\tau \in G$ ; in particular, we have

$$s = v\sigma(v)^{-1} = v\rho(v)^{-1}.$$

Let  $u = \sigma^2(v)^{-1}$ . The equations above yield

$$s = \sigma^2(u) \sigma(u)^{-1} \quad \text{and} \quad \rho(u) = u.$$

Therefore  $u \in K \otimes M$ , and this element satisfies the condition.

LEMMA 2.5. *The set  $\pi^{-1}(a \cdot F_s^\times)$  has 9 elements if it is non-empty.*

*Proof.* Suppose  $x_0 \in K \otimes F_s$  is such that  $x_0^3 \cdot F_s^\times = a \cdot F_s^\times$ . Then the map  $y \cdot F_s^\times \mapsto x_0 y \cdot F_s^\times$  defines a bijection between  $\pi^{-1}(1 \cdot F_s^\times)$  and  $\pi^{-1}(a \cdot F_s^\times)$ , so it suffices to show that  $|\pi^{-1}(1 \cdot F_s^\times)| = 9$ . Identify  $K \otimes F_s = F_s \times F_s \times F_s$ , and let  $\omega \in F_s^\times$  be a primitive cube root of unity. To simplify notation, write



$(z_1 : z_2 : z_3) = (z_1, z_2, z_3) \cdot F_s^\times$  for  $z_1, z_2, z_3 \in F_s$ . It is easy to check that  $\pi^{-1}(1 \cdot F_s^\times)$  consists of the following elements:

$$\begin{array}{lll} (1 : 1 : 1), & (1 : \omega : \omega^2), & (1 : \omega^2 : \omega), \\ (1 : 1 : \omega), & (1 : \omega : 1), & (\omega : 1 : 1), \\ (1 : 1 : \omega^2), & (1 : \omega^2 : 1), & (\omega^2 : 1 : 1). \end{array}$$

Each  $\xi \in \mathfrak{X}(M)$  extends uniquely to a  $K$ -algebra homomorphism

$$\widehat{\xi}: K \otimes_F M \rightarrow K \otimes_F F_s.$$

**THEOREM 2.6 (Rost).** *Let  $u \in (K \otimes M)^\times$  be such that  $\sigma^2(u)\sigma(u)^{-1} = s$ . The map  $\xi \mapsto \widehat{\xi}(u) \cdot F_s^\times$  defines an isomorphism of  $\Gamma$ -sets*

$$\Phi: \mathfrak{X}(M) \xrightarrow{\sim} \pi^{-1}(a \cdot F_s^\times).$$

*Proof.* If  $u \in (K \otimes M)^\times$  satisfies  $\sigma^2(u)\sigma(u)^{-1} = s$ , then

$$\sigma^2(u^3)\sigma(u^3)^{-1} = s^3 = \sigma^2(a)\sigma(a)^{-1},$$

so  $a^{-1}u^3$  is fixed under  $\sigma$ , hence  $a^{-1}u^3 \in M^\times$ . Therefore  $a^{-1}\widehat{\xi}(u)^3 \in F_s^\times$ , hence  $\widehat{\xi}(u) \cdot F_s^\times$  lies in  $\pi^{-1}(a \cdot F_s^\times)$ .

Note that the map  $\Phi$  does not depend on the choice of  $u$ : indeed,  $u$  is determined uniquely up to a factor in  $M^\times$ , and for  $m \in M^\times$  we have  $\widehat{\xi}(um) = \widehat{\xi}(u)\widehat{\xi}(m)$ , so  $\widehat{\xi}(um) \cdot F_s^\times = \widehat{\xi}(u) \cdot F_s^\times$ .

It is clear from the definition that the map  $\Phi$  is  $\Gamma$ -equivariant. Since  $|\mathfrak{X}(M)| = |\pi^{-1}(a \cdot F_s^\times)| = 9$ , it suffices to show that  $\Phi$  is injective to complete the proof. Extending scalars, we may assume that  $K \simeq F \times F \times F$ , and use the notation of Example 2.2. Then, up to a factor in  $M^\times$ , we have

$$\begin{aligned} u &= \sigma^2 \rho(s) e_{\text{Id}} + \sigma(s) e_\sigma + e_{\sigma^2} + \sigma(s) e_\rho + e_{\sigma\rho} + \sigma^2 \rho(s) e_{\sigma^2\rho} \\ &= \frac{r^2 t}{a_2} (e_{\text{Id}} + e_\rho) + r (e_\sigma + e_{\sigma^2\rho}) + (e_{\sigma^2} + e_{\sigma\rho}) \\ &= \left( \frac{r^2 t}{a_2}, r, 1 \right) \in K \otimes M = M \times M \times M. \end{aligned}$$

If  $\xi, \eta \in \mathfrak{X}(M)$  satisfy  $\widehat{\xi}(u) \cdot F_s^\times = \widehat{\eta}(u) \cdot F_s^\times$ , then  $\xi\left(\frac{r^2 t}{a_2}\right) = \eta\left(\frac{r^2 t}{a_2}\right)$  and  $\xi(r) = \eta(r)$ . Since  $M$  is generated by  $r$  and  $t$ , it follows that  $\xi = \eta$ .

REMARK 2.7. As pointed out by Rost [10], the map  $\pi$  factors through

$$W(F_s) = \{(\lambda, x) \cdot F_s^\times \mid \lambda^3 = N_K(x)\} \subseteq \mathbf{P}_{F \times K}(F_s) :$$

we have  $\pi = \pi_1 \circ \pi_2$ , where

$$\pi_2: \mathbf{P}_K(F_s) \rightarrow W(F_s), \quad x \cdot F_s^\times \mapsto (N_K(x), x^3) \cdot F_s^\times$$

and

$$\pi_1: W(F_s) \rightarrow \mathbf{P}_K(F_s), \quad (\lambda, x) \cdot F_s^\times \mapsto x \cdot F_s^\times.$$

There is a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}(M(K, a)) & \xrightarrow{\Phi} & \mathbf{P}_K(F_s) \\ \mathfrak{X}(i) \downarrow & & \downarrow \pi_2 \\ \mathfrak{X}(N(K, a)) & \xrightarrow{\Phi'} & W(F_s) \\ \downarrow & & \downarrow \pi_1 \\ \mathfrak{X}(F) & \xrightarrow{\Phi''} & \mathbf{P}_K(F_s), \end{array}$$

where  $\mathfrak{X}(i)$  is the map functorially associated to the inclusion

$$i: N(K, a) \hookrightarrow M(K, a)$$

and  $\Phi''$  maps the unique element of  $\mathfrak{X}(F)$  to  $a \cdot F_s^\times$ . The induced map  $\Phi'$  is an isomorphism of  $\Gamma$ -sets

$$\Phi': \mathfrak{X}(N(K, a)) \xrightarrow{\sim} \pi_1^{-1}(a \cdot F_s^\times).$$

### 3. INFLECTION POINT CONFIGURATIONS

Let  $V$  be a 3-dimensional vector space over  $F$ . Let  $S^3(V^*)$  be the third symmetric power of the dual space  $V^*$ , i.e., the space of cubic forms on  $V$ . A cubic form  $f \in S^3(V^*)$  is called *triangular* if its zero set in the projective plane  $\mathbf{P}_V(F_s)$  defines a triangle or, equivalently, if there exist linearly independent linear forms  $\varphi_1, \varphi_2, \varphi_3 \in V^* \otimes_F F_s$  such that  $f = \varphi_1 \varphi_2 \varphi_3$  in  $S^3(V^* \otimes F_s)$ . The *sides of the triangle* are the zero sets of  $\varphi_1, \varphi_2$ , and  $\varphi_3$ ; they form a 3-element  $\Gamma$ -set  $\mathfrak{S}(f)$ .

PROPOSITION 3.1. *Let  $f \in S^3(V^*)$  be a triangular cubic form and let  $K$  be the cubic étale  $F$ -algebra such that  $\mathfrak{X}(K) \simeq \mathfrak{S}(f)$ . Then we may identify the  $F$ -vector spaces  $V$  and  $K$  so as to identify  $f$  with a multiple of the norm form of  $K$ ,*

$$f = \lambda N_K \quad \text{for some } \lambda \in F^\times.$$

*In particular, the  $\Gamma$ -action on  $\mathfrak{S}(f)$  is trivial if and only if  $f$  factors into a product of three independent linear forms in  $V^*$ .*

*Proof.* Let  $f = \varphi_1 \varphi_2 \varphi_3$  for some linearly independent linear forms  $\varphi_1, \varphi_2, \varphi_3 \in V^* \otimes F_s$ . Since  ${}^\gamma \varphi_1 {}^\gamma \varphi_2 {}^\gamma \varphi_3 = \varphi_1 \varphi_2 \varphi_3$  for  $\gamma \in \Gamma$ , it follows by unique factorization in  $S^3(V^*)$  that there exist a permutation  $\pi_\gamma$  of  $\{1, 2, 3\}$  and scalars  $\lambda_{\pi_\gamma(i), \gamma} \in F_s^\times$  such that

$${}^\gamma \varphi_i = \lambda_{\pi_\gamma(i), \gamma} \varphi_{\pi_\gamma(i)} \quad \text{for } i = 1, 2, 3.$$

Since  ${}^{\gamma\delta} \varphi_i = {}^\gamma({}^\delta \varphi_i)$  for  $\gamma, \delta \in \Gamma$ , we have

$$\lambda_{\pi_{\gamma\delta}(i), \gamma\delta} \varphi_{\pi_{\gamma\delta}(i)} = \gamma(\lambda_{\pi_\delta(i), \delta}) \lambda_{\pi_\gamma \pi_\delta(i), \gamma} \varphi_{\pi_\gamma \pi_\delta(i)},$$

hence  $\pi_{\gamma\delta} = \pi_\gamma \pi_\delta$  and

$$(3.1) \quad \lambda_{\pi_{\gamma\delta}(i), \gamma\delta} = \gamma(\lambda_{\pi_\delta(i), \delta}) \lambda_{\pi_\gamma \pi_\delta(i), \gamma}.$$

The  $\Gamma$ -set  $\mathfrak{S}(f)$  is  $\{1, 2, 3\}$  with the  $\Gamma$ -action  $\gamma \mapsto \pi_\gamma$ ; therefore we may identify  $K$  with the  $F$ -algebra of  $\Gamma$ -equivariant maps

$$K = \text{Map}(\{1, 2, 3\}, F_s)^\Gamma.$$

For  $\gamma \in \Gamma$ , define  $a_\gamma \in \text{Map}(\{1, 2, 3\}, F_s^\times) = (K \otimes F_s)^\times$  by

$$a_\gamma(i) = \lambda_{i, \gamma}.$$

Clearly,  $a_\gamma = 1$  if  $\gamma$  fixes  $\varphi_1, \varphi_2$ , and  $\varphi_3$ ; moreover, by (3.1) we have  $a_\gamma {}^\gamma a_\delta = a_{\gamma\delta}$  for  $\gamma, \delta \in \Gamma$ , hence  $(a_\gamma)_{\gamma \in \Gamma}$  is a continuous 1-cocycle. By Hilbert's Theorem 90 [8, (29.2)], we have  $H^1(\Gamma, (K \otimes F_s)^\times) = 1$ , hence there exists  $b \in \text{Map}(\{1, 2, 3\}, F_s^\times)$  such that  $a_\gamma = b {}^\gamma b^{-1}$  for all  $\gamma \in \Gamma$ . For  $i = 1, 2, 3$ , let  $\psi_i = b(i) \varphi_i \in V^* \otimes F_s$ . Let also

$$\lambda = (b(1)b(2)b(3))^{-1}.$$

Computation shows that  ${}^\gamma \psi_i = \psi_{\pi_\gamma(i)}$  for  $\gamma \in \Gamma$  and  $i = 1, 2, 3$ , and  $f = \lambda \psi_1 \psi_2 \psi_3$  in  $S^3(V^* \otimes F_s)$ , hence  $\lambda \in F^\times$ . Define

$$\Theta: V \otimes F_s \rightarrow \text{Map}(\{1, 2, 3\}, F_s) = K \otimes F_s$$

by

$$\Theta(x): i \mapsto \psi_i(x) \quad \text{for } i = 1, 2, 3 \text{ and } x \in V \otimes F_s.$$

Since  $\psi_1, \psi_2, \psi_3$  are linearly independent,  $\Theta$  is an  $F_s$ -vector space isomorphism. It restricts to an isomorphism of  $F$ -vector spaces  $V \xrightarrow{\sim} K$  under which  $f$  is identified with  $\lambda N_K$ .

Now, let  $\mathfrak{I} \subseteq \mathbf{P}_V(F_s)$  be a 9-point set that has the characteristic property of the set of inflection points of a nonsingular cubic curve: the line through any two distinct points of  $\mathfrak{I}$  passes through exactly one third point of  $\mathfrak{I}$ . Let  $\mathfrak{L}$  be the set of lines in  $\mathbf{P}_V(F_s)$  that are incident to three points of  $\mathfrak{I}$ . This set has 12 elements, and  $\mathfrak{I}, \mathfrak{L}$  form an incidence geometry that is isomorphic to the affine plane over the field with three elements, see [7, §11.1]. In particular, there is a partition of  $\mathfrak{L}$  into four subsets  $\mathfrak{T}_0, \dots, \mathfrak{T}_3$  of three lines, which we call *triangles*, with the property that each point of  $\mathfrak{I}$  is incident to one and only one line of each triangle.

Assume  $\mathfrak{I}$  is stable under the action of  $\Gamma$ , and  $\Gamma$  preserves the triangle  $\mathfrak{T}_0$ . Let  $K$  be the cubic étale  $F$ -algebra whose  $\Gamma$ -set  $\mathfrak{X}(K)$  is isomorphic to  $\mathfrak{T}_0$ . By Proposition 3.1, we may identify  $V$  with  $K$  in such a way that the union of the lines in  $\mathfrak{T}_0$  is the zero set of the norm  $N_K$ .

**THEOREM 3.2.** *There exists  $a \in K^\times$  such that the  $\Gamma$ -set of vertices of the triangles  $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$  is  $\pi^{-1}(a \cdot F_s^\times)$ , where  $\pi: \mathbf{P}_K(F_s) \rightarrow \mathbf{P}_K(F_s)$  is defined in (2.1). The set  $\mathfrak{I}$  is the set of inflection points of the cubics in the pencil spanned by the forms  $T_K(a^{-1}X^3)$  and  $N_K(X)$ , and we have isomorphisms of  $\Gamma$ -sets*

$$\mathfrak{L} \simeq \mathfrak{X}(K) \coprod \mathfrak{X}(M(K, a)), \quad \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\} \simeq \mathfrak{X}(N(K, a)).$$

*Proof.* Fix an isomorphism  $K \otimes F_s \simeq F_s \times F_s \times F_s$ , and write simply  $(x_1 : x_2 : x_3)$  for  $(x_1, x_2, x_3) \cdot F_s^\times$ . The sides of  $\mathfrak{T}_0$  are then the lines with equation  $x_1 = 0$ ,  $x_2 = 0$ , and  $x_3 = 0$ . Let  $\mathfrak{I} = \{p_1, \dots, p_9\}$ . We label the points so that the incidence relations can be read from the representation of the affine plane over  $\mathbf{F}_3$  in Figure 1.

Say the line through  $p_1, p_2, p_3$  is  $x_1 = 0$ , and the line through  $p_4, p_5, p_6$  is  $x_2 = 0$ . We can then find  $u_1, u_2, u_3, v \in F_s^\times$  such that

$$p_i = (0 : u_i : 1) \quad \text{for } i = 1, 2, 3, \quad \text{and} \quad p_4 = (1 : 0 : v).$$



FIGURE 1

Since  $p_7$  lies at the intersection of  $x_3 = 0$  with the line through  $p_1$  and  $p_4$ , we have

$$p_7 = (1 : -u_1 v : 0).$$

Similarly,

$$p_8 = (1 : -u_2v : 0) \quad \text{and} \quad p_9 = (1 : -u_3v : 0).$$

Finally, since  $p_5$  (resp.  $p_6$ ) lies at the intersection of  $x_2 = 0$  with the line through  $p_1$  and  $p_8$  (resp.  $p_9$ ), we have

$$p_5 = (u_1 : 0 : u_2 v) \quad \text{and} \quad p_6 = (u_1 : 0 : u_3 v).$$

Collinearity of the points  $p_2, p_6, p_7$  (resp.  $p_2, p_5, p_9$  ; resp.  $p_3, p_6, p_8$ ) yields

$$u_1^2 = u_2 u_3, \quad (\text{resp. } u_2^2 = u_1 u_3; \text{ resp. } u_3^2 = u_1 u_2).$$

Therefore

$$u_1^3 = u_2^3 = u_3^3 = u_1 u_2 u_3.$$

Since  $u_1, u_2, u_3$  are pairwise distinct, it follows that there is a primitive cube root of unity  $\omega \in F_s$  such that

$$u_2 = \omega u_1 \quad \text{and} \quad u_3 = \omega^2 u_1.$$

Straightforward computations yield the vertices of the triangles  $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ :

$$\begin{aligned} \mathfrak{T}_1 : q_1 &= (1 : \omega^2 u_1 v : -v), & q'_1 &= (1 : u_1 v : -\omega^2 v), & q''_1 &= (\omega^2 : u_1 v : -v), \\ \mathfrak{T}_2 : q_2 &= (\omega : u_1 v : -v), & q'_2 &= (1 : u_1 v : -\omega v), & q''_2 &= (1 : \omega u_1 v : -v), \\ \mathfrak{T}_3 : q_3 &= (1 : \omega u_1 v : -\omega^2 v), & q'_3 &= (\omega^2 : \omega u_1 v : -v), & q''_3 &= (1 : u_1 v : -v). \end{aligned}$$

Let  $a_0 = (1, u_1^3 v^3, -v^3) \in (K \otimes F_s)^\times$ . It is readily verified that

$$\{q_1, q'_1, q''_1, q_2, q'_2, q''_2, q_3, q'_3, q''_3\} = \pi^{-1}(a_0 \cdot F_s^\times).$$

Since  $\mathfrak{I}$  is stable under the action of  $\Gamma$ , the point  $a_0 \cdot F_s^\times$  is fixed under  $\Gamma$ , hence for  $\gamma \in \Gamma$  there exists  $\lambda_\gamma \in F_s^\times$  such that

$$\gamma(a_0) = a_0 \lambda_\gamma \quad \text{in } K \otimes F_s.$$

Then  $(\lambda_\gamma)_{\gamma \in \Gamma}$  is a continuous 1-cocycle of  $\Gamma$  in  $F_s^\times$ . Hilbert's Theorem 90 yields an element  $\mu \in F_s^\times$  such that  $\lambda_\gamma = \mu \gamma(\mu)^{-1}$  for all  $\gamma \in \Gamma$ . Then for  $a = a_0 \mu$  we have  $a_0 \cdot F_s^\times = a \cdot F_s^\times$  and  $\gamma(a) = a$  for all  $\gamma \in \Gamma$ , hence  $a \in K^\times$ .

The inflection points of the cubics in the pencil spanned by  $\mathsf{T}_K(a^{-1}X^3)$  and  $\mathsf{N}_K(X)$  are the points  $(x_1 : x_2 : x_3)$  such that

$$\begin{cases} x_1^3 + (u_1 v)^{-3} x_2^3 - v^{-3} x_3^3 = 0, \\ x_1 x_2 x_3 = 0. \end{cases}$$

The solutions of this system are exactly the points  $p_1, \dots, p_9$ .

Finally, the  $\Gamma$ -set of sides of the triangle  $\mathfrak{T}_0$  is isomorphic to  $\mathfrak{X}(K)$  by hypothesis, and the map that associates to each side of a triangle its opposite vertex defines an isomorphism between the set of sides of  $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$  and the set  $\{q_1, \dots, q''_3\} = \pi^{-1}(a \cdot F_s^\times)$ . By Theorem 2.6, we have  $\pi^{-1}(a \cdot F_s^\times) \simeq \mathfrak{X}(M(K, a))$ , hence

$$\mathcal{L} \simeq \mathfrak{X}(K) \coprod \mathfrak{X}(M(K, a)).$$

This isomorphism induces an isomorphism

$$\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\} \simeq \mathfrak{X}(N(K, a)),$$

which can be made explicit by the following observation: the triangular cubic forms in the pencil spanned by  $T_K(a^{-1}X^3)$  and  $N_K(X)$  are the scalar multiples of  $N_K(X)$  (whose zero set is the triangle  $\mathfrak{T}_0$ ) and of  $T_K(a^{-1}X^3) - 3zN_K(X)$ , where  $z \in F_s^\times$  is such that  $z^3 = N_K(a^{-1})$ . The zero set of the latter form is  $\mathfrak{T}_1$ ,  $\mathfrak{T}_2$  or  $\mathfrak{T}_3$  depending on the choice of  $z$ , and the three values of  $z$  are in one-to-one correspondence with the elements in the fibre of the map  $\pi_1$  in Remark 2.7.

#### 4. NORMAL FORMS OF TERNARY CUBICS

Let  $V$  be a 3-dimensional vector space over  $F$  and let  $f \in S^3(V^*)$  be a nonsingular cubic form. Recall from the introduction the notation  $\mathfrak{I}(f)$  (resp.  $\mathfrak{L}(f)$ , resp.  $\mathfrak{T}(f)$ ) for the set of inflection points (resp. inflectional lines, resp. inflectional triangles) of  $f$ . The following result is a direct application of Theorem 3.2:

**COROLLARY 4.1.** *Let  $K$  be a cubic étale  $F$ -algebra. The following conditions are equivalent:*

- (i)  *$f$  is isometric to a cubic form  $T_K(a^{-1}X^3) - 3\lambda N_K(X)$  for some unit  $a \in K^\times$  and some scalar  $\lambda \in F$ ;*
- (ii)  *$\Gamma$  has a fixed point  $\mathfrak{T}_0 \in \mathfrak{T}(f)$  with  $\mathfrak{T}_0 \simeq \mathfrak{X}(K)$  (as  $\Gamma$ -sets of 3 elements).*

*When these conditions hold, we have*

$$\mathfrak{L}(f) \simeq \mathfrak{X}(K) \coprod \mathfrak{X}(M(K, a)) \quad \text{and} \quad \mathfrak{T}(f) \simeq \{\mathfrak{T}_0\} \coprod \mathfrak{X}(N(K, a)).$$

*Proof.* If  $f(X) = T_K(a^{-1}X^3) - 3\lambda N_K(X)$ , then computation shows that the zero set of  $N_K$  is an inflectional triangle of  $f$ . This triangle is clearly preserved under the  $\Gamma$ -action. Conversely, if  $\mathfrak{T}_0 \in \mathfrak{T}(f)$  is preserved under the  $\Gamma$ -action and  $K$  is the cubic étale  $F$ -algebra such that  $\mathfrak{X}(K) \simeq \mathfrak{T}_0$ , Theorem 3.2 yields an element  $a \in K^\times$  such that the forms  $T_K(a^{-1}X^3)$  and  $N_K(X)$  span the pencil of cubics whose set of inflection points is  $\mathfrak{I}(f)$ .

Applying Corollary 4.1 in the case where  $F$  is a finite field yields a direct proof of the following result from [7, p. 276]:

**COROLLARY 4.2.** *Suppose  $F$  is a finite field with  $q$  elements. For any nonsingular cubic form  $f$ , the number of inflectional triangles of  $f$  defined over  $F$  is 0, 1, or 4 if  $q \equiv 1 \pmod{3}$ ; it is 0 or 2 if  $q \equiv -1 \pmod{3}$ .*

*Proof.* Since  $F$  is finite, the action of  $\Gamma$  on  $\mathfrak{T}(f)$  factors through a cyclic group. If there is at least one fixed triangle  $\mathfrak{T}_0$ , then Corollary 4.1 yields a decomposition

$$\mathfrak{T}(f) \simeq \{\mathfrak{T}_0\} \coprod \mathfrak{X}(N(K, a)),$$

where  $N(K, a) = F[t]$  with  $t^3 = N_K(a)$ . If  $N(K, a)$  is a field, then it must be a cyclic extension of  $F$ , hence  $F$  contains a primitive cube root of unity and therefore  $q \equiv 1 \pmod{3}$ . Similarly, if  $N(K, a) \simeq F \times F \times F$ , then  $F$  contains a primitive cube root of unity. Thus, if  $q \equiv -1 \pmod{3}$ , the  $\Gamma$ -action on  $\mathfrak{T}(f)$  has either 0 or 2 fixed points. If  $q \equiv 1 \pmod{3}$  then  $F$  contains a primitive cube root of unity and either the polynomial  $x^3 - N_K(a)$  is irreducible or it splits into linear factors. Therefore the  $\Gamma$ -action on  $\mathfrak{T}(f)$  has either 0, 1 or 4 fixed points.

We next spell out the special case of Corollary 4.1 where the cubic étale  $F$ -algebra  $K$  is the split algebra  $F \times F \times F$ :

**COROLLARY 4.3.** *There is a basis of  $V$  in which  $f$  takes the generalized Hesse normal form  $a_1x_1^3 + a_2x_2^3 + a_3x_3^3 - 3\lambda x_1x_2x_3$  for some  $a_1, a_2, a_3 \in F^\times$  and  $\lambda \in F$  if and only if  $\Gamma$  has a fixed point  $\mathfrak{T}_0 \in \mathfrak{T}(f)$  and acts trivially on  $\mathfrak{T}_0$  (viewed as a 3-element subset of  $\mathfrak{L}(f)$ ).*

**EXAMPLE 4.4.** Let  $K$  be a cubic étale  $F$ -algebra and let  $f(X) = T_K(X^3)$ . By Corollary 4.1 we have

$$\mathfrak{L}(f) \simeq \mathfrak{X}(K) \coprod \mathfrak{X}(M(K, 1)) \quad \text{and} \quad \mathfrak{T}(f) \simeq \{\mathfrak{T}_0\} \coprod \mathfrak{X}(N(K, 1)).$$

The  $\Gamma$ -sets  $\mathfrak{X}(M(K, 1))$  and  $\mathfrak{X}(N(K, 1))$  are determined in Example 2.3:

$$\mathfrak{X}(M(K, 1)) \simeq \mathfrak{X}(F) \coprod \mathfrak{X}(D * F[\omega]) \coprod \mathfrak{X}(K \otimes F[\omega])$$

and

$$\mathfrak{X}(N(K, 1)) \simeq \mathfrak{X}(F) \coprod \mathfrak{X}(F[\omega]).$$

The map  $\mathfrak{X}(i): \mathfrak{X}(M(K, 1)) \rightarrow \mathfrak{X}(N(K, 1))$  functorially associated to the inclusion  $i: N(K, 1) \hookrightarrow M(K, 1)$  maps  $\mathfrak{X}(F) \coprod \mathfrak{X}(D * F[\omega])$  to  $\mathfrak{X}(F)$  and  $\mathfrak{X}(K \otimes F[\omega])$  to  $\mathfrak{X}(F[\omega])$ .

If  $K \simeq F \times F \times F$ , then  $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$  so  $f$  has a Hesse normal form. If  $K \not\simeq F \times F \times F$ , then the  $\Gamma$ -action on  $\mathfrak{X}(K)$ , hence also on  $\mathfrak{X}(K \otimes F[\omega])$ , is nontrivial. Therefore it follows from Corollary 4.3 that  $f$  has a generalized Hesse normal form over  $F$  if and only if the  $\Gamma$ -action on  $\mathfrak{X}(D * F[\omega])$  is trivial. This happens if and only if  $D \simeq F[\omega]$ , which is



equivalent to  $K \simeq F[\sqrt[3]{d}]$  for some  $d \in F^\times$ , by [8, (18.32)]. Indeed, for  $X = x_1 + x_2\sqrt[3]{d} + x_3\sqrt[3]{d^2}$ , computation yields

$$f(X) = 3(x_1^3 + dx_2^3 + d^2x_3^3 + 6dx_1x_2x_3).$$

Corollary 4.3 applies in particular when  $F$  is the field  $\mathbf{R}$  of real numbers:

**COROLLARY 4.5.** *Every nonsingular cubic form over  $\mathbf{R}$  can be reduced to a generalized Hesse normal form.*

*Proof.* It is clear from the Weierstrass normal form that every nonsingular cubic over  $\mathbf{R}$  has three real collinear inflection points, see [3, Prop. 14, p. 305]. The inflectional line through these points is fixed under  $\Gamma$ , hence the  $\Gamma$ -action on  $\mathfrak{T}(f)$  has at least one fixed point. The same argument as in Corollary 4.2 then shows that  $\Gamma$  has exactly two fixed points in  $\mathfrak{T}(f)$ . Let  $\mathfrak{T}_0, \mathfrak{T}_1 \in \mathfrak{T}(f)$  be the fixed inflectional triangles. Assume the  $\Gamma$ -action on  $\mathfrak{T}_0$  (viewed as a 3-element set) is not trivial, hence  $K \simeq \mathbf{R} \times \mathbf{C}$  in the notation of Corollary 4.1; we shall prove that the  $\Gamma$ -action on  $\mathfrak{T}_1$  is trivial. By Corollary 4.1, there is a unit  $a = (a_1, a_2) \in \mathbf{R} \times \mathbf{C}$  such that

$$\mathfrak{L}(f) \simeq \mathfrak{X}(\mathbf{R} \times \mathbf{C}) \coprod \mathfrak{X}(M(\mathbf{R} \times \mathbf{C}, a)).$$

By Theorem 2.6, we have an isomorphism of  $\Gamma$ -sets

$$\Phi: \mathfrak{X}(M(\mathbf{R} \times \mathbf{C}, a)) \xrightarrow{\sim} \pi^{-1}(a \cdot \mathbf{C}^\times) \subset \mathbf{P}_{\mathbf{R} \times \mathbf{C}}(\mathbf{C}).$$

We identify  $(\mathbf{R} \times \mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C}$  with  $\mathbf{C} \times \mathbf{C} \times \mathbf{C}$  by mapping  $(r, x) \otimes y$  to  $(ry, xy, \bar{x}y)$  for  $r \in \mathbf{R}$  and  $x, y \in \mathbf{C}$ . Then the  $\Gamma$ -action on  $\mathbf{P}_{\mathbf{R} \times \mathbf{C}} = \mathbf{P}_{\mathbf{C}}^3$  is such that the complex conjugation  $\bar{\phantom{x}}$  acts by

$$(x_1 : x_2 : x_3) \mapsto (\bar{x}_1 : \bar{x}_3 : \bar{x}_2).$$

If  $\xi \in \mathbf{R}$  and  $\eta \in \mathbf{C}$  satisfy  $\xi^3 = a_1$  and  $\eta^3 = a_2$ , and if  $\omega \in \mathbf{C}$  is a primitive cube root of unity, then the proof of Lemma 2.5 shows that  $\pi^{-1}(a \cdot \mathbf{C}^\times)$  consists of the following elements:

$$\begin{array}{lll} (\xi : \eta : \bar{\eta}), & (\xi : \omega\eta : \bar{\omega\eta}), & (\xi : \bar{\omega\eta} : \omega\bar{\eta}), \\ (\xi : \eta : \omega\bar{\eta}), & (\xi : \omega\eta : \bar{\eta}), & (\omega\xi : \eta : \bar{\eta}), \\ (\xi : \eta : \bar{\omega\eta}), & (\xi : \bar{\omega\eta} : \bar{\eta}), & (\bar{\omega}\xi : \eta : \bar{\eta}). \end{array}$$

The three points in the first row of this table are fixed under the  $\Gamma$ -action, whereas the  $\Gamma$ -action interchanges the second and third row. Therefore the first row corresponds to  $\mathfrak{T}_1$  under  $\Phi$ , and the proof is complete.

When the conditions in Corollary 4.1 do not hold, we may still consider the 4-dimensional étale  $F$ -algebra  $T(f)$  such that  $\mathfrak{X}(T(f)) = \mathfrak{T}(f)$ , and the 12-dimensional étale  $F$ -algebra  $L(f)$  such that  $\mathfrak{X}(L(f)) = \mathfrak{L}(f)$ , which is a cubic étale extension of  $T(f)$ . The separability idempotent  $e \in T(f) \otimes_F T(f)$  satisfies  $e \cdot (T(f) \otimes T(f)) \simeq T(f)$ , and hence yields a decomposition

$$T(f) \otimes_F T(f) \simeq T(f) \times T(f)_0$$

for some cubic algebra  $T(f)_0$  over  $T(f)$ . Likewise, multiplication in  $L(f)$  yields an isomorphism

$$e \cdot (L(f) \otimes T(f)) \simeq L(f);$$

hence

$$L(f) \otimes_F T(f) \simeq L(f) \times L(f)_0$$

for some cubic algebra  $L(f)_0$  over  $T(f)_0$ . By functoriality of the construction of  $L$  and  $T$ , the cubic form  $f_{T(f)}$  over  $V \otimes_F T(f)$  obtained from  $f$  by scalar extension to  $T(f)$  satisfies

$$L(f_{T(f)}) \simeq L(f) \otimes_F T(f) \quad \text{and} \quad T(f_{T(f)}) \simeq T(f) \otimes_F T(f).$$

Corollary 4.1 applied to  $f_{T(f)}$  shows that  $f_{T(f)}$  is isometric to

$$\top_{L(f)}(a^{-1}X^3) - 3\lambda \mathbf{N}_{L(f)}(X)$$

for some  $\lambda \in T(f)^\times$  and some  $a \in L(f)^\times$  such that  $L(f)_0$  is a Morley  $T(f)$ -algebra  $L(f)_0 \simeq M(L(f), a)$ .

## REFERENCES

- [1] BOURBAKI, N. *Algèbre. Chapitres 4 à 7*. Masson, Paris, 1981.
- [2] BRETAGNOLLE-NATHAN, J. Cubiques définies sur un corps de caractéristique quelconque. *Ann. Fac. Sci. Univ. Toulouse (4)* 22 (1958), 175–234.
- [3] BRIESKORN, E. and H. KNÖRRER. *Plane Algebraic Curves*. Translated from the German by J. Stillwell. Birkhäuser Verlag, Basel-Boston-Stuttgart, 1986.
- [4] BROWN, K. S. *Cohomology of Groups*. Graduate Texts in Mathematics 87. Springer-Verlag, New York, Heidelberg, Berlin, 1982.
- [5] HESSE, O. Über die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln. *J. Reine Angew. Math.* 28 (1844), 68–96.
- [6] ———. Algebraische Auflösung derjenigen Gleichungen 9ten Grades, deren Wurzeln die Eigenschaft haben, daß eine gegebene rationale und symmetrische Function  $\theta(x_\lambda, x_\mu)$  je zweier Wurzeln  $x_\lambda, x_\mu$  eine dritte Wurzel  $x_\kappa$  giebt, so daß gleichzeitig:  $x_\kappa = \theta(x_\lambda, x_\mu)$ ,  $x_\lambda = \theta(x_\mu, x_\kappa)$ ,  $x_\mu = \theta(x_\kappa, x_\lambda)$ . *J. Reine Angew. Math.* 34 (1847), 193–208.

- [7] HIRSCHFELD, J. W. P. *Projective Geometries over Finite Fields*. Oxford Mathematical Monographs. Clarendon Press, Oxford University Press, New York, 1979.
- [8] KNUS, M.-A., A. S. MERKURJEV, M. ROST and J.-P. TIGNOL. *The Book of Involutions*. American Mathematical Society Colloquium Publications 44. Amer. Math. Soc., Providence, RI, 1998.
- [9] KNUS, M.-A. and J.-P. TIGNOL. Quartic exercises. *Int. J. Math. Math. Sci.* 68 (2003), 4263–4323.
- [10] ROST, M. Notes on Morley's theorem. Private notes dated July 22, 2003.
- [11] WEBER, H. *Lehrbuch der Algebra, Bd. II*. F. Vieweg u. Sohn, Braunschweig, 1899.

(Reçu le 8 juin 2008)

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